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Perfect Domination Number and Perfect Bondage Number of Complete Grid Graph

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ABSTRACT
Grid graphs and domination are very important ideas in computer architecture and communication techniques. We present results about Perfect Domination Number and Perfect Bondage Number for Grid Graphs. We find Perfect dominating sets and Perfect domination number and Perfect bondage number for $G_{m,n}$ using special patterns.

Keywords: Grid graph, Complete grid graph, Domination number, Perfect Domination number, Perfect Bondage number.

1.INTRODUCTION

Starting in the eighties domination numbers of Cartesian products were intensively investigated. In the meantime, some papers on domination numbers of cardinal products of graph was initiated by Vizing[10]. He conjectured that the domination number of the Cartesian product of two graph is always greater than or equal to the product of the domination numbers of the two factors. This conjecture is still unproven. For complete grid graphs, i.e. graphs $P_k \times P_n$, algorithms were given for a fixed k which compute $\gamma(P_k \times P_n)$ in O(n) time [7]. In fact, the domination number problem for $k \times n$ grids, where k is fixed, has a constant time solution. In this paper we present a survey of Perfect domination numbers and Perfect bondage numbers of complete grid graphs $P_k \times P_n$, K=2,3,4,5.

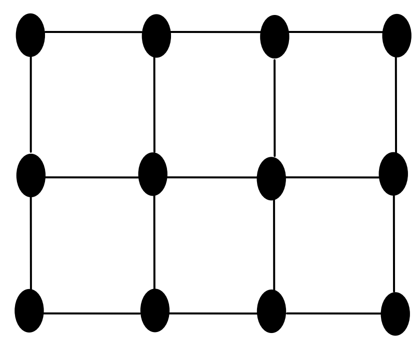
2. DEFINITIONS

For notations and graph theory terminology, we follow Bondy and Murthy[1].

Let G=(V,E) be a simple graph with vertex set V and edge set E. A subset D of V is a dominating set of G if every vertex of V \ D is adjacent to at least one vertex of D. The minimum cardinality of a dominating set is called the domination number of G which is denoted as $\gamma(G)$.

A set D of vertices of a graph G is a Perfect dominating set if for every vertex u of V \ D , $|N(u) \cap D| = 1$. A Perfect dominating set of minimum cardinality is a minimum Perfect dominating set and its cardinality is Perfect domination number of G denoted by $\gamma_P(G)$.

The Perfect bondage number is minimum number of edges whose removal from original graph increase Perfect domination number in resultant graph denoted by $\beta_P(G)$.



COMPLETE GRID GRAPH $G_{3,4}$

A two-dimensional complete grid graph is an $m \times n$ graph $G_{m,n} = P_m \times P_n$, the product of path graphs on m and n vertices. The Cartesian products of paths P_m and P_n with disjoint sets of vertices V_m and V_n and edge sets E_m and E_n is the graph with vertex set $V(P_m \times P_n)$ and edge set $E(P_m \times P_n)$ such that $((g_1, h_1),(g_2, h_2)) \in E(P_m \times P_n)$, if and only if either $g_1 = g_2 \ \ and \ \ (h_1, h_2) \in E(P_n) \ \ or \ \ h_1 = h_2 \ \ and \ \ (g_1, g_2) \in E(P_m)$. One example of a complete grid graph $G_{3,4} = P_3 \times P_4$ is shown in figure 1.

A grid graph $G_{m,n}$ has mn nodes and (m-1)n+(n-1)m=2mn-m-n edges. We observe that the path graph

$P_n = G_{1,n} = G_{n,1}$ and cycle graph $C_4 = G_{2,2}$. From the definition of complete grid graph $P_k \times P_n$ we observe that for k=1 the grid graph is nothing but path graph that is $P_1 \times P_n = P_n \times P_1 = P_n$.

H-MERGE AND V-MERGE OPERATIONS:

From the definition of complete grid graph $P_k \times P_n$ we observe that for k=n=2 the grid graph $P_2 \times P_2$ is a cycle on 4 vertices.

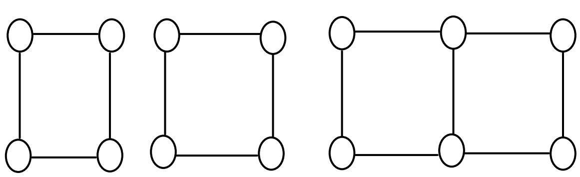
H-Merging of two cycles

Let cycle C_1 and C_2 be two cycles on 4 vertices. Let cycle C_1 have vertex set $\{u_1, u_2, u_3, u_4\}$ and cycle C_2 have vertex set $\{v_1, v_2, v_3, v_4\}$ then vertex set of new graph obtained by H-merging denoted as $C_1{}^\wedge C_2$ is $\{u_1, u_2 = v_1, u_3, u_4 = v_3, v_2, v_4\}$. Edges in $C_1{}^\wedge C_2$ includes all the edges of C_1 and C_2 with $(u_2, u_4) = (v_1, v_3)$

This gives that

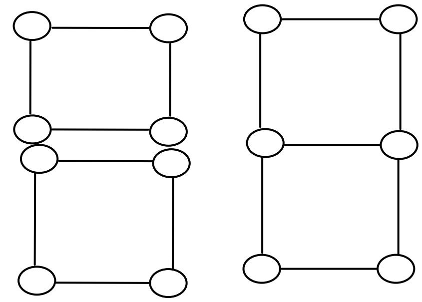
$$|V(C_1{}^\wedge C_2)| = |V(C_1)| + |V(C_2)| - 2 \quad \text{and}$$
$$|E(C_1{}^\wedge C_2)| = |E(C_1)| + |E(C_2)| - 1$$

Thus H-Merging of two cycles gives a complete grid graph $G_{2,2}{}^\wedge G_{2,2} = P_2 \times P_3 = G_{2,3}$



V-Merging of two cycles

In similar way we define another operation V-Merging which gives a complete grid graph shown as below.



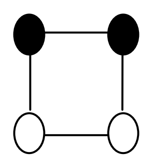
Lemma : 1

The Perfect domination number of grid graph $G_{2,n} = P_2 \times P_n$ *for* $n \geq 2$ *is*

$$\Upsilon_P(G_{2,n}) = \begin{cases} \left[\dfrac{n+1}{2}\right] & n = even \\ \left[\dfrac{n}{2}\right] & n = odd \end{cases}$$

Proof

Step : 1 Result is true for n = 2 $\Upsilon_P(G_{2,2}) = 2$



Step: 2 Suppose result is true for n = k, So,

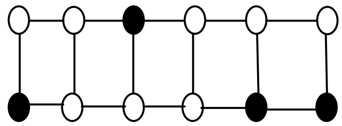
$$\Upsilon_P(G_{2,k}) = \begin{cases} \left[\dfrac{k+1}{2}\right] & k = even \\ \left[\dfrac{k}{2}\right] & k = odd \end{cases}$$

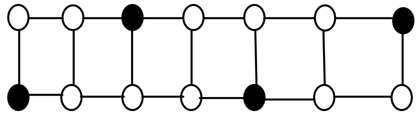
Step : 3 If n = k + 1

CASE A: k=even so, $\Upsilon_P(P_2 \times P_k) = \left[\dfrac{k+1}{2}\right]$

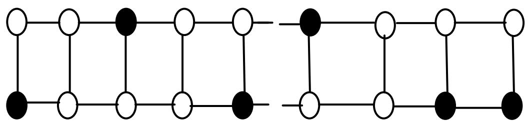
n=k+1 is odd

To prove: $\Upsilon_P(P_2 \times P_{k+1}) = \left[\dfrac{k+1}{2}\right]$

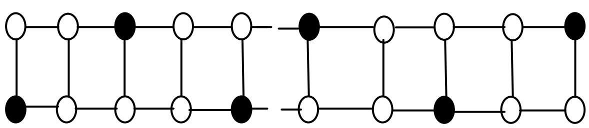




Consider $G_{2,k}$ .Let D be Perfect dominating set in $G_{2,k}$ .k=even So, In D we will have two adjacent vertices in last square given below.



To construct $G_{2,k+1}$. We have to add 2 vertices(1 pair) in $G_{2,k}$ .Now to perfect dominate $P_2 \times P_{k+1}$, we will do re-arrangement of D and with same number of vertices, we can perfect dominate $P_2 \times P_{k+1}$. See the figure given below.



So, $\Upsilon_P(P_2 \times P_{k+1}) = \left[\frac{k+1}{2}\right]$

CASE:B k=odd

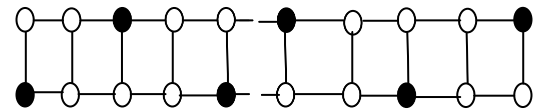
$$\Upsilon_P(P_2 \times P_k) = \left[\frac{k}{2}\right]$$

n=k+1 is even

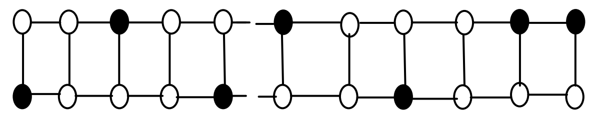
To prove: $\Upsilon_P(P_2 \times P_{k+1}) = \left[\frac{k+1}{2}\right]$

Consier $P_2 \times P_k$, k=odd. Let D be perfect dominating set. We can observe that every vertex $\notin$D is adjacent to only one

vertex $\in$D.see figure given below.



Now we add one pair in $P_2 \times P_k$ to construct $P_2 \times P_{k+1}$. In $P_2 \times P_k$, last pair is having one vertex $\in$D. So to perfect dominate $P_2 \times P_{k+1}$, we will have to add one more vertex.

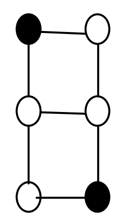


So, $\Upsilon_P(P_2 \times P_{k+1}) = \left[\frac{k}{2}\right] + 1 = \left[\frac{k+1}{2}\right]$

Lemma 2 The Perfect domination number of complete grid graph $G_{3,n} = P_3 \times P_n$ *for* $n \geq 2$ is $\Upsilon_P(P_3 \times P_n) = n$

Proof:

Step : 1 Result is true for n = 2 $\Upsilon_P(P_3 \times P_2) = 2$



Step: 2 Suppose result is true for n = k, So,

$$\Upsilon_P(P_3 \times P_k) = k$$

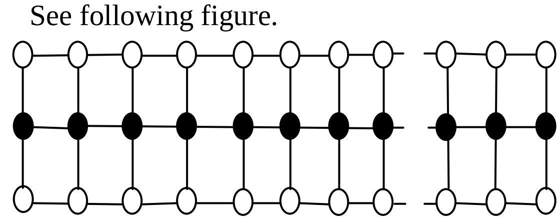
Step : 3 If n = k + 1

To Prove: $\Upsilon_P(P_3 \times P_{k+1}) = k+1$

Consider $P_3 \times P_k$ and $\Upsilon_P(P_3 \times P_k) = k$

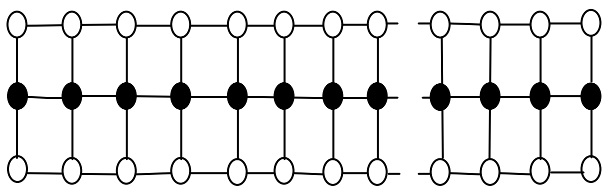
Let D be perfect dominating set with |D|=k

See following figure.



To construct P_{k+1}, we have to add $P_3 \times P_1$ at last pair. By figure given below

$$(P_3 \times P_{k+1}) = (P_3 \times P_k) \cup (P_3 \times P_1)$$
$$\Upsilon_P(P_3 \times P_{k+1}) = \Upsilon_P(P_3 \times P_k) + \Upsilon_P(P_3 \times P_1)$$

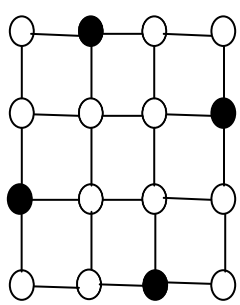


Lemma 3:- The Perfect domination number of grid graph $P_5 \times P_n$ for $n \geq 4$ is 2n-4

Proof :

Step:-1 Result is true for n=4

$$\Upsilon_2(P_4 \times P_4) = 4$$



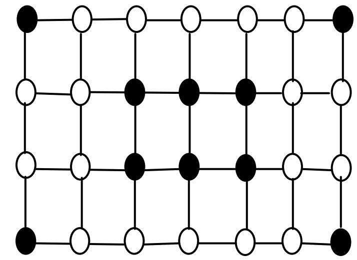
Step:-2 Suppose result is true for n=k

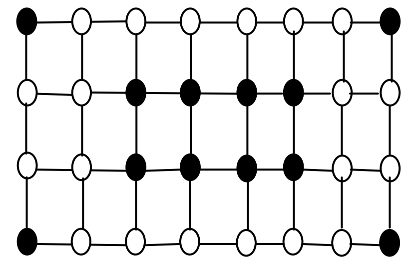
$$\Upsilon_2(P_4 \times P_k) = 2k - 4$$

Step:-3 Check result for n=k+1

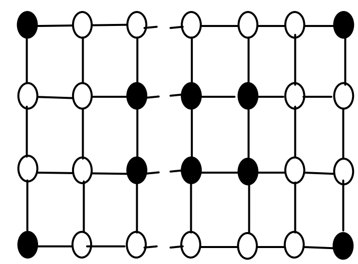
To prove: $\Upsilon_2(P_4 \times P_{k+1}) = 2(k+1) - 4$

Consider one example of $P_4 \times P_7$ & $P_4 \times P_8$



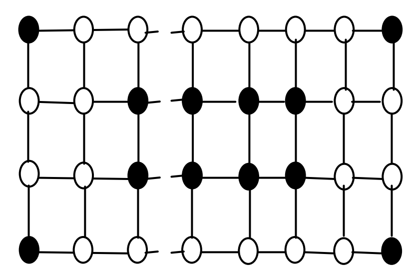


Now consider in general form $P_4 \times P_k$



Let D be Perfect dominating set in $P_4 \times P_k$ with Cardinality 2K-4. Now consider $P_4 \times P_{k+1}$

To Construct $P_4 \times P_{k+1}$ we add one path $P_4 \times P_1$ at the end.



To dominate $P_4 \times P_{k+1}$, we have to add two more vertices in D (Observing pattern)

$\Upsilon_P(P_4 \times P_{k+1})$ =(2k-4)+2=2k-2=2(k+1)-4

Lemma 4: The Perfect domination number of grid graphs $P_5 \times P_n$ for $k \geq 3$ is

$$\frac{5n}{3} \quad ; \quad n = 3k$$
$$5\left(\frac{n+2}{3}\right) \quad ; n = 3k+1$$
$$5\left(\frac{n+1}{3}\right) \quad ; \ n = 3k+2$$

Proof:

Case 1: if n=3k

Now divide the $5 \times n$ grid graphs into k blocks $P_5 \times P_3$ each. Extending the above result for $P_5 \times P_{3k}$ where each of the k blocks $P_5 \times P_3$ contribute 5 vertices into the minimal Perfect dominating set. We get 5k vertices into minimal Perfect dominating Set. Using the minimality of Perfect dominating set of $G_{5,3}$, we claim the minimality of Perfect dominating set of $P_5 \times P_{3k}$.

Hence $\Upsilon_P(P_5 \times P_{3k}) = 5k$

Case 2: if n=3k+1

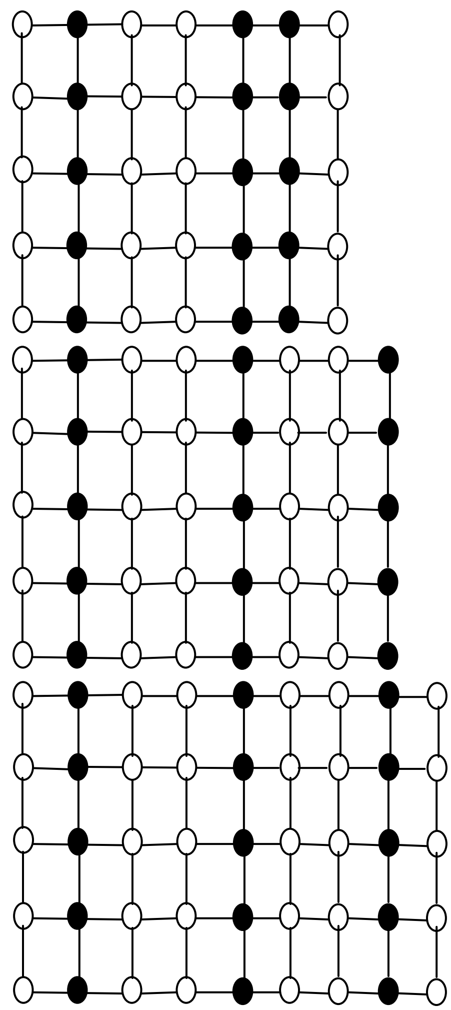
We divide the $5 \times n$ grid graphs into k blocks of $P_5 \times P_3$ each and a path $P_5 \times P_1$. As in case 1 we get patterns in the k blocks of $P_5 \times P_3$ giving 5 vertices into the Perfect dominating set from each block for Perfect domination of the vertices in the block $P_5 \times P_1$, We need 5 vertices to be added to Perfect dominating set. Thus the Perfect dominating set of $P_5 \times P_{3k+1}$ has 5k+5 vertices. As in case 1 we claim that this is the minimal Perfect dominating set of $P_5 \times P_{3k+1}$.

Hence
$$\Upsilon_P(P_5 \times P_{3k+1}) = \Upsilon_P(P_5 \times P_{3k}) + \Upsilon_P(P_5) = 5k+5$$
$$= 5(k+1)$$

Case 3: if n=3k+2

We divide the $5 \times n$ grid graphs into k blocks of $P_5 \times P_3$ each and a path $P_5 \times P_2$. As in case 1 we get patterns in the k blocks of $P_5 \times P_3$ giving 5 vertices into the Perfect dominating set from each block for Perfect domination of the vertices in the block $P_5 \times P_2$, We need 5 vertices to be added to Perfect dominating set. Thus the Perfect dominating set of $P_5 \times P_{3k+2}$ has 5k+5 vertices. As in case 1 we claim that this is the minimal Perfect dominating set of $P_5 \times P_{3k+2}$.

Hence
$$\Upsilon_P(P_5 \times P_{3k+2}) = \Upsilon_P(P_5 \times P_{3k}) + \Upsilon_P(P_5 \times P_2) = 5k+5$$
$$= 5(k+1)$$
(Refer following figures.)



Lemma 5: The Perfect bondage number of complete grid graph $G_{2,n} = P_2 \times P_n$ for $n \geq 5$ is

$$\beta_P(P_2 \times P_n) = \begin{cases} 1, & n = odd \\ 2, & n = even \end{cases}$$

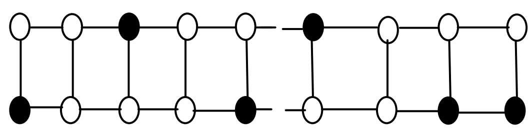
Proof:

Consider the complete grid graph $P_2 \times P_n$ .By lemma 1perfect domination number , $G_{2,n} = P_2 \times P_n \;\; for\; n \geq 2 \;\; is$

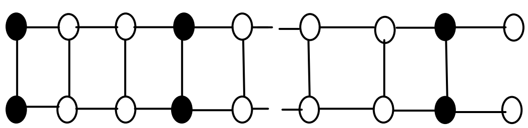
$$\Upsilon_P(G_{2,n}) = \begin{cases} \left\lceil \dfrac{n+1}{2} \right\rceil & n = even \\ \left\lceil \dfrac{n}{2} \right\rceil & n = odd \end{cases}$$

Case 1. Let n=even

Let D be perfect dominating set in this graph.. Then by Lemma 1,



Now if we remove an edge between last pair then in resultant graph , Let S is perfect dominating Set. (see following pattern)

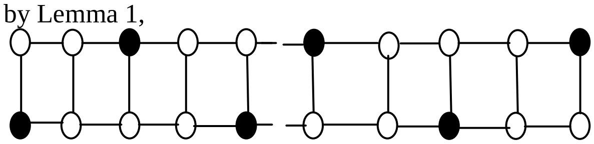


$|S| = |D| + 1$

So, Perfect Bondage number for this graph is 1.

Case 2 Let n=odd

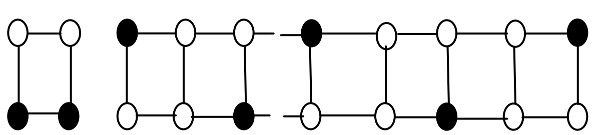
Let D be perfect dominating set in this graph.. Then by Lemma 1,



Now if we remove two edges from graph such that last square becomes isolate from graph and let S be perfect dominating set in resultant graph then

$|S| = |D| + 1$

(see following pattern)



So, Perfect Bondage number for this graph is 2.

Lemma 6

The Perfect bondage number of complete grid graph

$$G_{3,n} = P_3 \times P_n = \begin{cases} 2 \;, & otherwise \\ 1 \;, & n = 5,1 \end{cases}$$

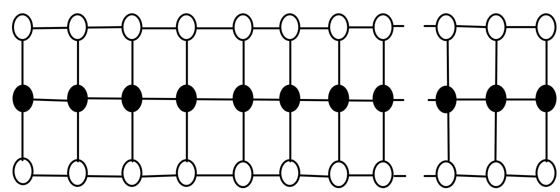
Proof:

By Lemma 2,

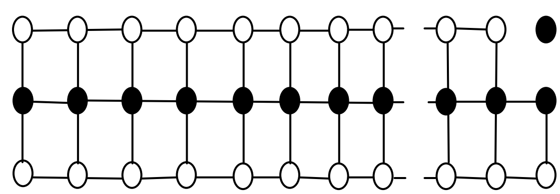
The Perfect domination number of complete grid graph $G_{3,n} = P_3 \times P_n \;\; for\; n \geq 2$ is

$\Upsilon_P(P_3 \times P_n) = n$

Let D be Perfect Dominating Set in graph.Observe following figure.



We can see that all corner vertices are not in D. If we make any corner vertex isolate by removing its two adjacent edges then Let S be perfect dominating set in resultant graph. See following figure.



We can See that

$|S| = |D| + 1$

So, Perfect Bondage number for this graph is 2.

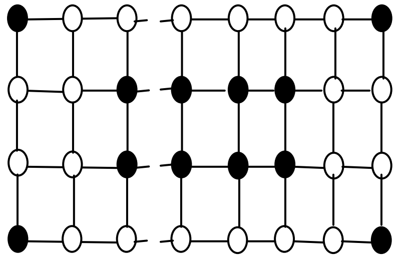
Lemma: 7 The Perfect bondage number of complete grid graph $G_{4,n} = P_4 \times P_n$ is 2 $n \geq 2$

Proof:

Consider the complete grid graph $G_{4,n} = P_4 \times P_n$

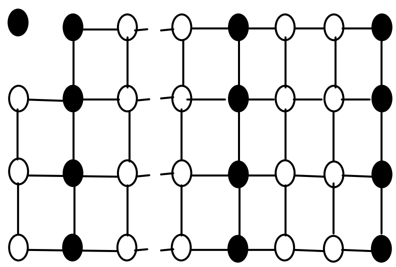
By lemma 3, $\Upsilon_P(P_4 \times P_n) = 2n - 4$.

Now consider in general form $P_4 \times P_k$



Let D be perfect dominating set in this graph.

Now if we remove two edges which are adjacent to corner vertex to make corner vertex isolated,then Let S be perfect dominating set in resultant graph.



Then |D| > |S|

If n=3k+1 then |S|=|D|+3

Otherwise |S|=|D|+1

Here there exist not any other perfect dominating set having cardinality minimum then S.

So, Perfect bondage number of this graph is 2.

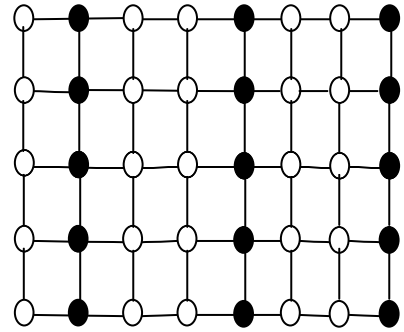
Lemma :8

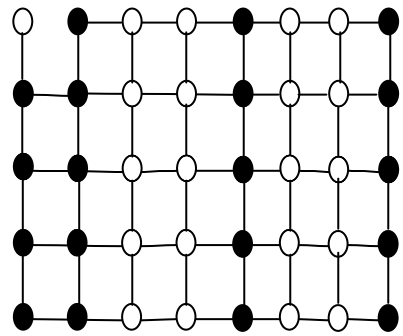
The perfect bondage number of complete grid graph $G_{5,n} = P_5 \times P_n = 1, \quad n \geq 5$

Proof:

Consider the complete grid graph $P_5 \times P_n$.

For example $P_5 \times P_4$ and $P_5 \times P_5$





By lemma 4,

The Perfect domination number of grid graphs $P_5 \times P_n$ for $k \geq 3$ is

$$\begin{cases} \frac{5n}{3} & ; \quad n = 3k \\ 5\left(\frac{n+2}{3}\right) & ; n = 3k+1 \\ 5\left(\frac{n+1}{3}\right) & ; \ n = 3k+2 \end{cases}$$

Let D be perfect Dominating set in $P_5 \times P_n$.
Consider corner vertex if we remove an edge which is adjacent to any corner vertex $\notin$ D and having another end vertex $\in$ D.
Then perfect Dominating number of resultant graph will increase.

Graph	Domination Number	Perfect Domination Number	Perfect Bondage Number
	$\Upsilon(P_k \times P_n)$	$\Upsilon_P(P_k \times P_n)$	$\beta_P(P_k \times P_n)$
$P_2 \times P_n$	$\left\lfloor \frac{n+2}{2} \right\rfloor, n \geq 1$	$\left\lceil \frac{n+1}{2} \right\rceil \quad n = even$ $\left\lceil \frac{n}{2} \right\rceil \quad n = odd$	1 if n=odd 2 if n=even
$P_3 \times P_n$	$\left\lfloor \frac{3n+4}{4} \right\rfloor, n \geq$	n	1 if n=5,1 2 otherwise
$P_4 \times P_n$	n+1 n=1,2,3,5, 6,9 n otherwise for $n \geq 1$	$n \geq 4$ is 2n-4	2
$P_5 \times P_n$	$\left\lfloor \frac{6n+6}{5} \right\rfloor$ $n = 2, 3, 7$ $\left\lfloor \frac{6n+8}{5} \right\rfloor$ $otherwise\ n \geq 1$	$k \geq 3$ $\frac{5n}{3} \quad ; \quad n = 3k$ $5\left(\frac{n+2}{3}\right) \quad ; n = 3k+1$ $5\left(\frac{n+1}{3}\right) \quad ; \ n = 3k+2$	1

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