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Parameter Estimation for the Exponential distribution model Using Least-Squares Methods and applying Optimization Methods

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Abstract: We find parameter estimates of the Exponential distribution models using least-squares estimation method for the case when partial derivatives were not available, the Nelder and Meads, and Hooke and Jeeves optimization methodswere used and for the case when first partial derivatives are available, the Quasi – Newton Method (Davidon-Fletcher-Powel (DFP) and the Broyden-Fletcher-Goldfarb-Shanno (BFGS) optimization methods)were applied. The medical data sets of 21Leukemiacancer patients with time span of 35 weeks ([3],[6]) were used.

Keywords: Exponential Distribution model, Nelder and Meads, and Hooke and Jeeves, DFP and BFGS optimization methods, Parameter estimation, Least Square method, Kaplan-Meier estimates, Survival rate Estimates, Variance-Covariance

I. Introduction

The method of linear least-squares requires that a straight line be fitted to a set of data points such that the sum of squares of the vertical deviations from the points to be minimized ([17]).

Adrien Merie Legendre(1752-1833) is generally credited for creating the basic ideas of the method of least squares. Some people believe that the method was discovered at the same time by Karl F. Gauss (1777-1855), Pierre S. Laplace (1749-1827) and others. Furthermore, Markov's name is also included for further development of these ideas. In recent years, ([17],[18])an effort have been made to find better methods of fitting curves or equations to data, but the least-squares method remained dominant, and is used as one of the important methods of estimating the parameters. The least-squares method ([18], [19]) consists of finding those parametersthat minimize a particular objective function based on squared deviations.

It is to be noted that for the least-squares estimation method, we are interested to minimize some function of the residual, that is, we want to find the best possible agreement between the observed and the estimated values. To define the objective function *F*, we set up a vector of residuals $r_i = y_i^{obs} - y_i^{est}, i = 1,2,\cdots,m$.(1.1)

Then the objective function is a sum of squared residuals - the term 'least-squares' derives from this kind of function:

$$F = \sum_{i=1}^{m} r_i^2 = \sum_{i=1}^{m} (y_i^{obs} - y_i^{est})^2 \ . \tag{1.2}$$

The objective function is the sum of the squares of the deviations between the observed values and the corresponding estimated values ([17],[18]). The maximum absolute discrepancy between observed and estimated values is minimized using optimization methods.

We treated Kaplan-Meier estimates $(KM(t_i))$ ([1],[5]) as the observed values (y_i^{obs}) of the objective functionand the survivor rate estimates $(S(t_i))$ of exponential distribution models as the estimated value (y_i^{est}) of the objective function*F* ([5]) . We considered the objective function for the models of the form

$$F = \sum_{i=1}^{m} f_i \, (KM\,(t_i) - S\,(t_i))^2 \tag{1.3}$$

where f_i is the number of failures at time t_i and *m* is the number of failure groups.

We used the procedure as follows:

- Note that the Kaplan-Meier method is independent of parameters, so for a particular value of time t_i we find the value of the Kaplan-Meier estimate $KM(t_i)$ of the survival function.
- We suppose that the survivor function of Exponential model at time t_i is $S(t_i;a)$, and with the starting value of the parameters a_0, we can find the value of the survivor function $S(t_i;a_0)$.
- From the numerical values of the Kaplan-Meier estimates $KM(t_i)$, and the survivor function $S(t_i;a)$ of theexponential model at time t_i, we can evaluate errors $|S(t_i;a) - KM(t_i)|$.
- The function value with a suitable starting point a_0 is given by $F(t_i;a_0) = \max_i |S(t_i;a_0) - KM(t_i)|$.
- We have numerical value of the function at initial point a_0 and this function value can be used in numerical optimizationsearch methods to find the minimum point a^* (optimal value of the parameter a).

1. Exponential Distribution Model

The exponential distribution ([3],[4]) is a very commonly used distribution in reliability and life testing. The single-parameter exponential *pdf* is

$$f(t) = \lambda \exp(-\lambda t), \quad t \geq 0, \lambda > 0 \tag{2.1}$$

The reliability (or survivor) function of the exponential distribution is

$$S(t) = 1 - F(t) = 1 - \int_0^t f(x)dx \tag{2.2}$$

Or $$S(t) = \exp(-\lambda t). \tag{2.3}$$

$$H(t) = \frac{f(t)}{S(t)} = \lambda \tag{2.4}$$

where λ parameter is the constant failure-rate, (or hazard rate). Since the hazard rate is constant, therefore it is a useful model for lifetime data where used items are to be considered as good-as-new ones.

II. Exponential Distribution Models using Least-Squares Methods and Applying Nelder and Meads and Hook and Jeeves Search Methods

For a practical application of the least-squares estimation method, when partial derivatives of the objective function are not available, we considered the data of twenty-one leukemia patients. Nelder and Meads ([10],[14],[15]) and Hook and Jeeves ([7],[8]) are simplex methods and are useful for optimizing the nonlinear programming problems. These are numerical methods without calculating the derivatives of the objective function. These methods do not require first partial derivatives (gradients) so may converge very slow or even may diverge at all ([8], [9]). The numerical results of Exponential distribution model using Nelder and Mead's and Hooke and Jeeves search methods have been presents in this paper. The results include function values, parameter estimates, survivor-rate estimates, Kaplan-Meier estimates ([1], [[5]) and otherinformation.

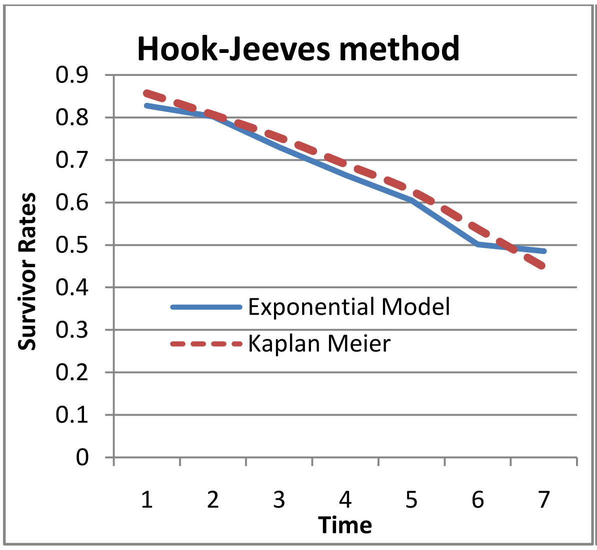
Table 1: Comparison of Survival Rate estimates

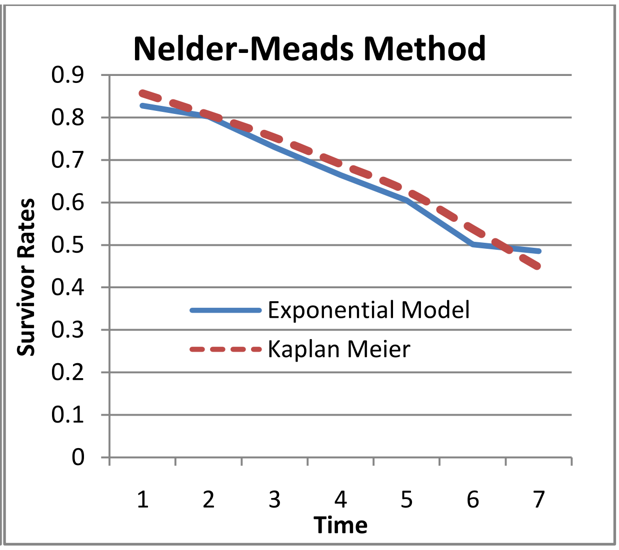
Failure Time (Weeks)	Number of Failures	**Nelder and Meads**		**Hook and Jeeves**	
		Exponential Model	Kaplan Meier	Exponential Model	Kaplan Meier
6	3	0.8281428977	0.85714285714	0.8280900288	0.85714285714
7	1	0.8025205496	0.8067226890	0.8024607779	0.8067226890
10	1	0.7303126154	0.7529411764	0.7302349115	0.7529411764
13	1	0.6646016934	0.6901960784	0.6645097687	0.6901960784
16	1	0.6048032056	0.6274509803	0.6047002489	0.6274509803

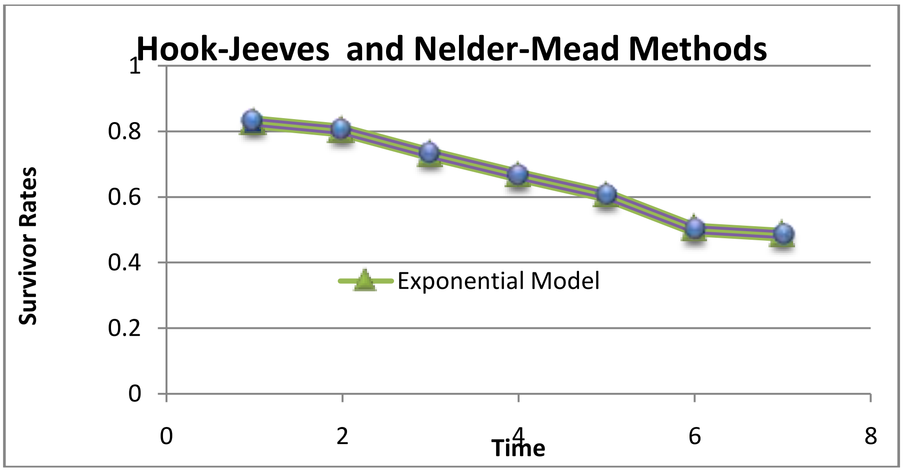
22	1	0.5008634792	0.5378151260	0.5007462466	0.5378151260
23	1	0.485367001	0.4481792717	0.4852482322	0.4481792717

Table 2:

	Nelder and Meads	Hook and Jeeves
Estimates of Parameter	3.1428259611129838E-02	3.1438899999998537E-02
Optimal Functional value	3.7187729316226603E-03	3.7068960496299763E-03







III. Exponential Distribution Models using Least-Squares Methods and ApplyingQuasi Newton Methods

In order to find the parameter estimates of Exponential probability distribution models, we used the Davidon-Fletcher-Powel (DFP) and the Broyden-Fletcher-Reeves-Shanno (BFGS) optimization methods ([21], [23], [25]).

To apply these optimization methods, we need to find the first partial derivatives of the objective function *F* of eq. (1.3).

$$\frac{\partial F}{\partial \lambda} = 2\sum_{i=1}^{m} f_i \left(S(t_i) - KM(t_i)\right)\frac{\partial S(t_i)}{\partial \lambda} \tag{4.1}$$

where $$\frac{\partial S(t)}{\partial \lambda} = -tS(t) \ . \quad (4.2)$$

Now using eq.(4. 1) and eq.(4.2) and eq. (1.3) in the DFP and in the BFGS optimization method, we find the estimated value of the parameter $\hat{\lambda}$ and other information ([25], [26], [27]).

Table 3: Numerical Results using Quasi –Newton Method

	DFP	BFGS
Estimates of Parameter($\hat{\lambda}$)	2.96462923265E-2	2.964389273494E-2
Optimal Functional value(*F*)	5.05544097205E-03	5.05544162231E-03
Gradient at $\hat{\lambda}$	9.7664E-08	-1.2641E-05
The Variance-Covariance at $\hat{\lambda}$	8.13899E-04	8.13962E-04

IV. Conclusion

The Survival rate estimates for the 21 Leukemia patients for the period of 35 week under observations were compared using Kaplan Meier estimation and exponential distribution model. We found that the results (like the parameter estimates) for the exponential distribution model were approximately samefor both the cases when the derivatives of an objective function were not available (Using the Hook and Jeeves, and Nelder and Meads method) and when first partial derivatives of the objective function were available (using Quasi-Newton method (DFP and BFGS methods).

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References

[1]. Kaplan E.L., P.Meier P. (1958) Nonparametric estimation from incomplete observations ,J. Amer. Statist. Assoc., 53, pp. 457-481.
[2]. Freireich et al. (1963"The Effect of 6-mercaptopurine on the duration of steroid induced remission in acute leukemia", Blood 21), pp. 699-716.
[3]. Bain, L.J.,Englehardt, M. (1991) Statistical Analysis of Reliability and Life-Testing Models: Theory and Methods, 2, Marcel Dekker.
[4]. Lawless J.F. (1982) Statistical Models and Methods for life-time Data, John Wiley and Sons.
[5]. Cox D.R.,Oaks D. (1984) Analysis of Survival Data. London Chapman and Hall.
[6]. Robert Hooke and T.A. Jeeves, Westinghouse Research Laboratories, Piitsburgh, Pennsylvania, "Direct Research Solution of Numerical and statistical Problem research", International Journal of Statistics, Vol.16, pp.670-683,1960.
[7]. S. Babu, T. S. Kumar and V. Balasubramanian, "Optimizing Pulsed Current Gas Tugsten Arc Welding Parameters of AA6061 Aluminium Alloy Using Hooke and Jeeves Algorithm," Transactions of Nonferrous Metals Society of China, Vol. 18, No. 5, 2008, pp. 1028-1036.
[8]. G.S Kirat , A.N Surde "Review of Hooke and Jeeves Direct Search Solution Method AnalysisApplicable To Mechanical Design Engineering" Novateur Publications International Journal of Innovations in Engineering Research and Technology [IJIERT], ISSN: 2394-3696, Volume 1, Issue 2, Dec-2014.
[9]. J.A. Nelder, R. Mead, "A Simplex method for function minimization", Computer Journal 7, (1965), 308-313.
[10]. K.I.M. McKinnon, "Convergence of the Nelder-Mead simplex method to a nonstationary point", SIAM Journal on Optimization, 9, (1998), 148- 158.
[11]. C.J. Price, I.D. Coope, D. Byatt, "A convergent variant of the Nelder-Mead algorithm", Journal of Optimization Theory and Applications, 113, (2002), 5–19.
[12]. L. Nazareth, P. Tseng, "Gilding the Lily: A Variant of the Nelder-Mead Algorithm Based on Golden-Section Search", Comput. Optim. Appl, vol. 22, no. 1, pp. 133–144, 2002.
[13]. L. Han, M. Newmann, "Effect of dimensionality on the Nelder-Mead simplex method",Optimization Methods and Software, 21, 1(2006), 1-16.

[14]. F. Gao, L. Han, “Implementing the Nelder-Mead Simplex Algorithm with Adaptive Parameters”, Comput. Optim. Appl, vol., no. , pp. , 4th May 2010.
[15]. Nam P., Bogdan M.Wilamow “improved Nelder Mead’s Simplex method and Applications”, Journal Of Computing, Volume 3, Issue 3, ISSN 2151-9617, March 2011.
[16]. K. Levenberg, “A Method for the Solution of Certain Problems in Least Squares,” Quart. Appl. Mach., vol. 2, pp. 164–168, 1944.
[17]. Plackett, R.L. (1972). “The discovery of the method of least squares. Biometrika”, 59,239–251.
[18]. J. T. Betts, “Solving the Nonlinear Least Square Problems: Application of a General Method”, Journal of Optimization Theory and Applications, vol. 8, no. 4, 1976.
[19]. Harper H.L. (1974–1976). “The method of least squares and some alternatives”. Part I, II, II, IV, V, VI. International Satistical Review, 42, 147–174; 42, 235–264; 43, 1–44; 43, 125–190; 43, 269–272; 44, 113–159.
[20]. Davidon, W .C . : Variable Metric Method for Minimization. ANL-5990 (rev.) , Argonne National Laboratory, Nov. 1959. 21.
[21]. Fletcher, R. ; and Powell, M. J. D . : A Rapidly Convergent Descent Method for Minimization. Computer J. , vol. 6, Apr . 1963-Jan. 1964, pp . 163-168.
[22]. Broyden, C. G. (1970), "The convergence of a class of double-rank minimization algorithms", Journal of the Institute of Mathematics and Its Applications, 6: 76–90, doi:10.1093/imamat/6.1.76
[23]. Fletcher, R. (1970), "A New Approach to Variable Metric Algorithms", Computer Journal, 13 (3): 317–322, doi:10.1093/comjnl/13.3.317
[24]. Fletcher, Roger (1987), Practical methods of optimization (2nd ed.), New York: John Wiley & Sons, ISBN 978-0-471-91547-8
[25]. Goldfarb, D. (1970), "A Family of Variable Metric Updates Derived by Variational Means", Mathematics of Computation, 24 (109): 23–26, doi:10.1090/S0025-5718-1970-0258249-6
[26]. Shanno, David F.; Kettler, Paul C. (July 1970), "Optimal conditioning of quasi-Newton methods", Mathematics of Computation, 24 (111): 657–664, doi:10.1090/S0025-5718-1970-0274030-6.