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Analysis of Nonlinear Vibrations of Single Walled Carbon Nanotubes

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ABSTRACT: *This paper provides a comparative study of the variational iteration method (VIM), Adomian decomposition method (ADM) and new homotopy perturbation method. These methods are very powerful and efficient techniques for solving different kinds of linear and nonlinear differential equations arising in different fields of science and engineering. Our results are also compared with numerical simulations. An excellent agreement is observed between the series solutions obtained by these three methods and a numerical solution.*

Keywords: *Adomian decomposition method, variational iteration method, new homotopy perturbation method, nonlinear vibration, carbon nanotube.*

I. INTRODUCTION

Most of the scientific problems in fluid mechanics are modeled by nonlinear differential equations. It is well known that exact solutions of these nonlinear boundary value problems are difficult to obtain when non-Newtonian fluids are taken into account. Therefore, numerical and analytical methods are used to handle these type of problems. However, the numerical methods are comparatively tedious and difficult due to their stability and convergence problems. In the recent decade, many different analytic methods have been introduced to solve the nonlinear problems, such as the homotopy analysis method (HAM) [1], the variational iteration method (VIM) [7-12], the new homotopy perturbation method (NHPM) [13,14], and the Adomian decomposition method (ADM) [15-20]. A great deal of comparison of various analytical techniques with numerical works has been done (Table 1).

In this study, we have applied VIM, NHPM and ADM to find the approximate solutions to the problem of the nonlinear vibrations of single walled carbon nanotubes embedded in an elastic medium. Very recently, Siddiqui and Farooq [2] have provided a comparison of variational iteration and Adomian decomposition methods in solving nonlinear thin film flow problems. These methods generate the solution in convergent series with components that are elegantly computed. Furthermore, these analytic methods avoid the complexities provided by other pure numerical methods [3, 4 ,5]. The results reveal that the proposed analytical methods provide an effective mathematical tool to handle a large class of linear and nonlinear differential equations in engineering and chemical sciences.

Nomenclature and Units:

Symbols	Name	Unit	Experimental values
$[\omega]$	Nonlinear free vibration Frequency	cm^{-1}	$18\,cm^{-1}$
$[w_b]=\sqrt{w_l^{\,2}+w_k^{\,2}}$	linear vibration frequency	THz	$0.138\,THz$
$[w_k]=\sqrt{k/\rho A}$	Dimensionless	None	None
$[w_l]=\pi^2/l^2\sqrt{EI/\rho A}$	Dimensionless	None	None
$[k]$	Spring constant	N/m^2	$10^7\,N/m^2$
$[X]$	Dimensionless vibration amplitude	None	None
$[\alpha]$	Dimensionless	None	0.25
$[\tau]=\omega t$	dimensionless time	None	None
$[x]=W/r$	Dimensionless vibration of SWNT	None	None

$[\rho]$	Density	kg/m^3	$1.3*10^3 kg/m^3$
$[A]$	Cross sectional area	$(nm)^2$	$2.7759\,(nm)^2$
$[E]$	Young modulus	TPa	$1.1TPa$
$[I]$	Cross sectional interia moment	TPa	$1.32TPa - 1.58TPa$
$[W]$	Vibration of SWNT	nm	$0.94nm$
$[r] = \sqrt{I/A}$	Dimensionless	nm	$0.94nm$
$[t]$	Time	s	None
$[l]$	Length of SWNT	nm	$45nm$

II. Mathematical Formulation of The Problem

Consider the SWNT of length *l*, Young's modulus *E*, density ρ, cross-sectional area *A*, and cross-sectional inertia moment *I*, embedded in an elastic medium with material constant *k*. The nonlinear vibration equation for these carbon nanotubes (CNT) is in the following form [1]:

$$\frac{d^2W}{dt^2} + \left(\frac{\pi^4 EI}{\rho A l^4} + \frac{k}{\rho A}\right)W + \frac{\pi^4 E}{4\rho l^4}W^3 = 0, \tag{1}$$

Using the following dimensionless variables

$$r = \sqrt{\frac{I}{A}}, x = \frac{W}{r}, W_l = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}, w_k = \sqrt{\frac{k}{\rho A}} \tag{2}$$

the equation (1) can be transformed to the following dimensionless nonlinear vibration equation.

$$\frac{d^2x}{d\tau^2} + ax + bx^3 = 0, \tag{3}$$

$$where\ a = \frac{w_b^{\ 2}}{\omega^2},\ b = \frac{\alpha w_l^{\ 2}}{\omega^2} \tag{4}$$

in which $\alpha = 0.25$ and $w_b = \sqrt{w_l^{\ 2} + w_k^{\ 2}}$, is the linear free vibration frequency.

The initial and boundary conditions are

$$x(0) = X,\quad \dot{x}(0) = 0 \tag{5}$$

2.1 Solution by Variational Iteration Method

To illustrate the basic idea of He's variational iteration method [7-9], we consider the following nonlinear functional equation:

$$Lx(\tau) + Nx(\tau) = g(\tau) \tag{6}$$

where $Lx(\tau)$ is a linear operator, $Nx(\tau)$ a nonlinear operator and $g(\tau)$ an inhomogeneous term. Inokuti et al.[6] suggested a method of general Lagrange multiplier . Then, we can construct a correct functional as follows:

$$x_{n+1}(\tau) = x_n(\tau) + \int_0^\tau \lambda(s)\left(Lx_n(\tau) + N\tilde{x}_n(\tau) - g(s)\right)ds, \tag{7}$$

where $\lambda(s)$ is a Lagrange multiplier that can be identified optimally via the variational theory [7-9]. The subscript *n* denotes the *n*th approximation, and $x_n(\tau)$ is considered to be restricted variation, that is, $\delta\tilde{x}_n(\tau) = 0$. In this method the Lagrange multiplier $\lambda(s)$ is first determined optimally. The successive approximation $x_{n+1}(\tau), n \geq 0,$ of the solution $x(\tau)$ can be readily obtained by using this determined Lagrange multiplier with any selective function $x_0(\tau)$. Consequently, the solution is given by $x(\tau) = \lim_{n\to\infty} x_n(\tau)$. For the convergence criteria and error estimates of the VIM we refer the reader to [7-12]. According to the variational iteration method, we can construct correction functional of Eqn.(3) as follows:

$$x_{n+1} = x_n + \int_0^\tau \lambda(s)(x_n^{''}(s) + ax_n(s) + bx_n^{\ 3}(s))ds \tag{8}$$

With $\lambda(s) = s - \tau$, we start with the initial guess $x_0(\tau) = X$ in the above iteration formula and obtain the following approximate solutions:

$$x_0(\tau) = X \tag{9}$$

$$x_1(\tau) = X - (\frac{\tau^2}{2})(aX + bX^3) \tag{10}$$

$$x_2(\tau) = X - \left(4aX + b^3X^3\right)\frac{\tau^2}{2} - \left(\frac{(a+bX^2)\left(aX - 3b^3X^3\right)}{24}\right)\tau^4 - \left(\frac{(b^3X^3)(a+bX^2)^2}{40}\right)\tau^6 + \left(\frac{(b^3X^3)(a+bX^2)^3}{448}\right)\tau^8 \tag{11}$$

By considering the three iteration, we get

$$x(\tau) \approx x_2(\tau) \tag{12}$$

2.2 Solution by new homotopy perturbation method

Perturbation method depends on small parameter and chooses unsuitable small parameter can be lead to wrong solution. Homotopy is an important part of topology [13, 14] and it can convert any non-linear problem in to a finite linear problems and it doesn't depend on small parameter [13,14]. For introducing the homotopy perturbation method, we consider to nonlinear problem with boundary condition in the following form:

$$A(x) - f(\tau) = 0,\ \tau \in \Omega$$

$$B\left(x, \frac{\partial x}{\partial \tau}\right) = 0,\ x \in \Gamma \tag{13}$$

where A is a general differential operator, B is a boundary operator, $f(\tau)$ is a known analytic function and Γ is the boundary of the domain Ω . The operator A can be divided into two operators L and N , where L and N are linear and non-linear operators sequentially. So, we can write Eqn. (13) as the following form:

$$L(x) + N(x) - f(\tau) = 0 \tag{14}$$

Homotopy perturbation $H(v, p)$ may be written as follows:

$$H : \Omega \times [0,1] \to R \tag{15}$$

$$H(v, p) = N(v) + (p-1)N(x_0) = 0, \tag{16}$$

where embedding parameter $p \in [0,1], \tau \in \Omega, v$ is an approximation of x, x_0 is an initial approximation of x .

Now we introduce a new NHPM:

$$H(v, p) = N(v) + (p-1)N(u_0) = 0, \tag{17}$$

In other words we assume only non-linear operator and $p \in [0,1]$.

The normalized nonlinear vibration Eqn. (3) defines the initial value problem. The new homotopy perturbation method is used to give the approximate solutions of the non-linear Eqn.(3). To find the solution of Eqn. (3), we construct the Homotopy as follows:

$$(1-p)\left[\frac{d^2x}{d\tau^2} + ax + b\,x(0)^3\right] + p\left[\frac{d^2x}{d\tau^2} + ax + bx^3\right] = 0 \tag{18}$$

The approximate analytical solution of Eqn. (3) is

$$x = x_0 + px_1 + p^2x_2 + \ldots \tag{19}$$

Substituting Eqn.(19) in Eqn.(18) and comparing the coefficients of like powers of p in Eqn.(18) we get,

$$p^0 : \frac{d^2x_0}{d\tau^2} + ax_0 + bX^3 = 0 \tag{20}$$

$$p^1 : \frac{d^2x_1}{d\tau^2} + ax_1 = -(b{x_0}^3) \tag{21}$$

The initial and boundary approximations are as follows:

$$x(0) = X,\ \dot{x}(0) = 0\ , \tag{22}$$

$$x_1(0) = 0,\ \dot{x}_1(0) = 0 \tag{23}$$

Solving the Eqn. (20) using the initial condition Eqn. (22), we obtain the following result:

$$x_0(\tau) = \frac{-b}{a} + \left(\frac{b}{a} + X\right)\cos(\sqrt{a}\,\tau) \tag{24}$$

Since more number of terms in $x_1(\tau)$ when solving the Eqn.(21), we considering the first iteration alone. Hence we get,

$$x(\tau) \approx x_0(\tau) \tag{25}$$

2.3 Solution by Adomian Decomposition Method

A detailed description of the Adomian decomposition method is given in [15-20]. Here, we convey only the basic steps as a reminder. Writing Eqn.(3) in operator form, we obtain

$$Lx(\tau) + Rx(\tau) + Nx(\tau) = g(\tau), \tag{26}$$

where L is the highest order derivative which is assumed to be easily invertible. That is, $Lx(\tau) = \frac{dx}{d\tau}, Rx(\tau) = 0, Nx(\tau) = ax + bx^3$ represents the nonlinear term and $g(x) = 0$ is the source term. According to the Adomian method the solution $x(\tau)$ can be expanded into an infinite series

$$x(\tau) = \sum_{n=0}^{\infty} x_n(\tau), \tag{27}$$

where the components $x_n(\tau)$ are usually determined recursively. The nonlinear term $Nx(\tau)$ can be decomposed into infinite polynomials given by

$$Nx(\tau) = \sum_{n=0}^{\infty} A_n(x_0, x_1, x_2, ..., x_n), \tag{28}$$

where A_n are the so-called Adomian polynomials of $x_0, x_1, x_2, \ldots, x_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\right]_{\lambda=0}, n = 0,1,2,.. \tag{29}$$

or equivalently,

$$A_0 = N(x_0), \tag{30}$$

$$A_1 = x_1 N'(x_0), \quad A_2 = x_2 N'(x_0) + \frac{1}{2}x_1^2 N''(x_0). \tag{31}$$

It is well known that these polynomials can be constructed for all classes of nonlinearity according to algorithm set by Adomian [15-20]. The general algorithm of the decomposition method for the nonlinear system (3), yields the recurrence relation,

$$x_0(\tau) = A \tag{32}$$

$$x_{n+1}(\tau) = -L^{-1}(A_n), n = 0,1,2,... \tag{33}$$

where A is a constant of integration and can be found from the boundary condition (5). The first few terms of the Adomian polynomials $A_n, n \geq 0,$ for this problem are given by:

$$A_0 = -(aX + bX^3), \tag{34}$$

$$A_1 = (aX + bX^3)(a + 3bX^2)(\frac{\tau^2}{2}), \tag{35}$$

From these above results, we obtain the following components:

$$x_0(\tau) = X, \tag{36}$$

$$x_1(\tau) = -(aX + bX^3)(\frac{\tau^2}{2}) \tag{37}$$

$$x_2(\tau) = (aX + bX^3)(a + 3bX^2)(\frac{\tau^4}{24}) \tag{38}$$

By considering the three iteration,we get,

$$x(\tau) \approx x_0(\tau) + x_1(\tau) + x_2(\tau) \tag{39}$$

III. ANALYTICAL EXPRESSION OF NONLINEAR VIBRATION

III. The dimensionless non linear Eqn. (3) defines the initial value problem. The variational iteration method [7-12], new homotopy perturbation method [13-14], and Adomian decomposition method [15-20], are used to give the approximate solutions of the non-linear Eqn. (3). Using these three methods, we can obtain the analytical expressions of vibration. Using VIM the analytical expression of vibration can be obtained as,

$$x(\tau) = X - (4aX + b^3X^3)\frac{\tau^2}{2} - \frac{(a+bX^2)}{24}(aX - 3b^3X^3)\tau^4 - \left(\frac{(b^3X^3)(a+bX^2)^2}{40}\right)\tau^6 + \left(\frac{(b^3X^3)(a+bX^2)^3}{448}\right)\tau^8 \quad (40)$$

The analytical expression of vibration using NHPM is,

$$x(\tau) = \frac{-b}{a} + (\frac{b}{a} + X)\cos(\sqrt{a}\,\tau) \quad (41)$$

The analytical expression of vibration using ADM is,

$$x(\tau) = X - \frac{\tau^2}{2}(aX + bX^3) + \frac{\tau^4}{24}(aX + bX^3)(a + 3bX^2) \quad (42)$$

IV. NUMERICAL SIMULATION

The non-linear differential equation (3) for the given initial – boundary conditions is being solved numerically. The function pdex, in two dimensional differential equation solver and Grapher V 1.0 software which is a function of solving the initial – boundary value problems for non-linear ordinary differential equations is used to solve this equation. The analytical expressions of vibration $x(\tau) = W/r$ are compared with simulation results for various time τ and kinetic parameters in the Tables (1-9). Satisfactory agreement is noted.

V. DISCUSSION AND CONCLUSIONS

Eqns. (40-42) are the new and simple analytical expressions of dimensionless vibration of SWNT for all values of parameter. The variation of vibration of SWNT with respect to dimensionless time for various values of parameter is represented in Figs. 1-4. From the figures it is observed that the dimensionless vibration of SWNT decreases absurdly from its initial value for all values of parameter a and b. In this study, we have illustrated how VIM, NHPM and ADM can be employed to obtain the approximate analytical solutions of a nonlinear initial value problem in single walled carbon nanotubes. It is concluded that this methods are very powerful and efficient techniques for solving different kinds of nonlinear problems arising in various fields of science and engineering. In all the methods, the error percentage is less than 4.66. However the in some places error percentage in one method is higher than other two methods because of initial conditions and the coefficients in the nonlinear differential equations.

Also VIM requires the evaluation of the Lagrangian multiplier λ, where as ADM requires the evaluation of the Adomian polynomials that mostly require tedious algebraic calculations. ADM requires the use of Adomian polynomials for nonlinear terms, and this need more work. Three iterations are used to get the analytical results by using ADM and VIM. But using NHPM the obtained analytical result is in one iteration. Therefore the accuracy of the methods is not only depends upon the iteration, it also depends upon the parameter involved in the differential equations and the initial/boundary conditions of the given problems.

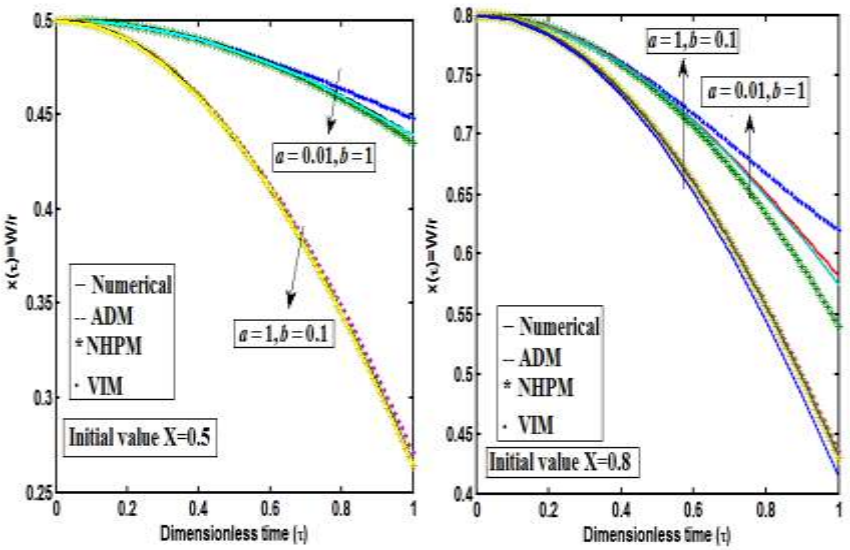

Fig.1. Plot of dimensionless vibration $x(\tau)$ versus dimensionless time τ for various values of the parameter, a, b and initial values X=0.5 and X=0.8 using Eqns. (40-42).

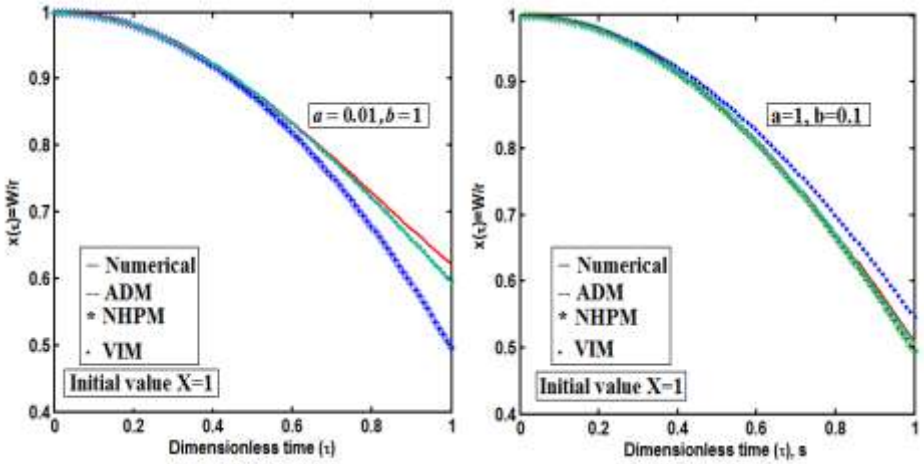

Fig. 2. Plot of dimensionless vibration $x(\tau)$ versus dimensionless time τ for various values of the parameter, a,b and initial value X=1 using Eqns. (40-42).

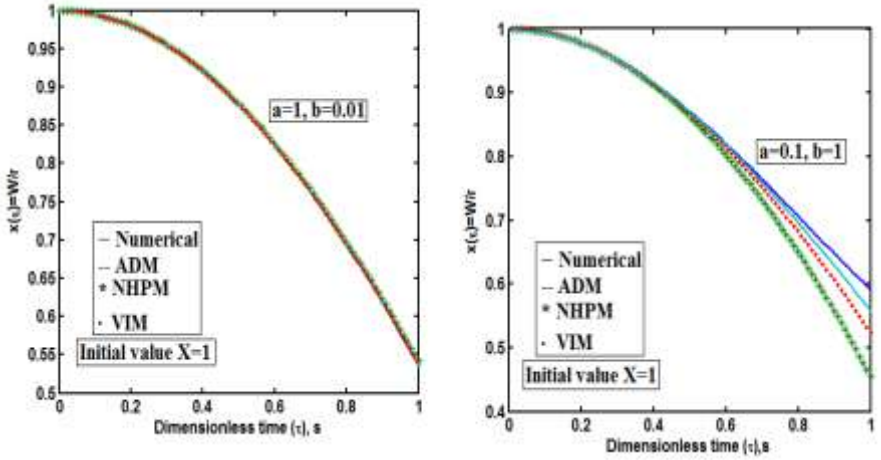

Fig. 3. Plot of dimensionless vibration $x(\tau)$ versus dimensionless time τ for various value of the parameter, a,b and initial value X=1 using Eqns. (40-42).

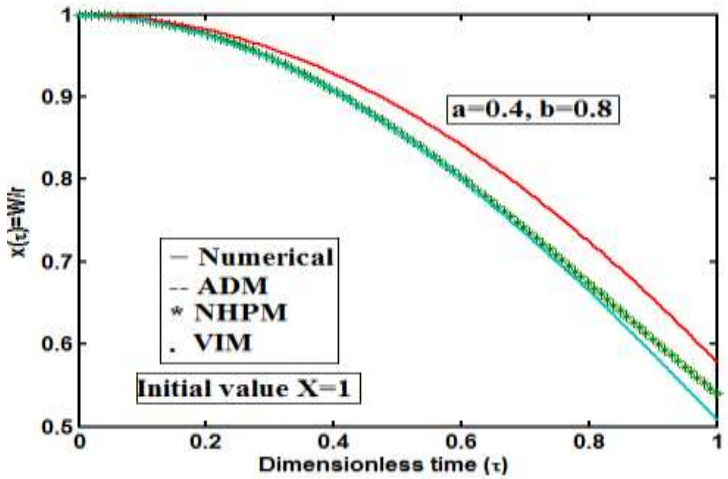

Fig. 4. Plot of dimensionless vibration $x(\tau)$ versus dimensionless time τ for various value of the parameter, a,b and initial value X=1 using Eqns. (40-42).

Table 1. Recent contributions to the theoretical modeling of the comparison of analytical and Numerical solutions of two or three methods.

Publication		*Name of Equation*	*Modeling method (HPM, HAM, ADM, etc.)*
Author(s)	*Reference*		
A.M. Wazwaz	Journal of Computational and Applied Mathematics 207 (2007) 129 – 136	Homogeneous and nonhomogeneous advection problem	Adomian decomposition method (ADM) and variational iteration method (VIM)
D. D. Ganji *et al.*	International Journal of Science & Technology 2 (2) (2007) 179-188	Generalized Hirota-Satsuma coupled KdV equation	Variational iteration method (VIM) and Adomian's decomposition method (ADM)
M.S.H. Chowdhury *et al.*	Communications in Nonlinear Science and Numerical Simulation 14 (2009) 371–378	The temperature distribution along a straight fin equation	Homotopy perturbation method (HPM) and homotopy analysis method (HAM)
D.D. Ganji *et al.*	Walailak J Sci & Tech; 11 (7) (2014) 593-609.	Three-Dimensional Flow of a Walter's B Fluid in Vertical Channel	Adomian decomposition method (ADM), homotopy perturbation method (HPM) and variational iteration method (VIM)
Elaf Jaafar Ali *et al.*	Journal of Basrah Researches (Sciences) 37 (2011) 1817-2695	Extraordinary differential equation	Homotopy perturbation method (HPM) and variational iteration method (VIM)
E.Babolian *et al.*	Journal of applied sciences, (2012) 1812-5654	Nonlinear differential equations	Homotopy perturbation method (HPM) and Adomian decomposition method (ADM)
Jamshad Ahmad *et al.*	International Journal of Basic Sciences & Applied Research.3 (3) (2014) 173-179,	Cubic nonlinear Schrödinger equation	Laplace decomposition method(LDM) and Adomian decomposition method (ADM)
K.R. Raslan1 *et al.*	Gen. Math. Notes, 20 (2) (2014) 125-135 2219-7184	Improved modified Kortweg-de varies equation.	Adomian decomposition method (ADM) and variational iteration method (VIM)
Malihe Bagheri *et al.*	The Journal of Mathematics and Computer Science 5 (4) (2012) 288-296	Volterra and Feredholm integral equations	Differential transform method(DTM) and Homotopy perturbation method (HPM)
Muhammad Shakil *et al.*	International Journal of Research in Applied, Natural and Social Sciences (IMPACT: IJRANSS) 1 (3) (2013) 37-48	Nonlinear wave equation, Klein-Gordon equation	Adomian decomposition method (ADM) and homotopy perturbation method (HPM)
A.M.Siddiqui *et al.*	Applied Mathematical Sciences, 6 (99) (2012) 4911 - 4919	Thin film flow	Adomian decomposition method (ADM) and variational iteration method (VIM)
N. Shawagfeh et al.	Applied Mathematics Letters 17 (2004) 323-328.	Ordinary differential equations.	Runge-Kutta method and Adomian decomposition method (ADM)
S. Abbasbandy *et al.*	Applied Mathematics and Computation 173 (1) (2006) 493–500	Non-singular integral equations	Homotopy perturbation method (HPM) and Adomian decomposition method (ADM)
S.Abbasbandy *et al.*	Chaos, Solitons & Fractals 31 (1) (2007) 257–260	Blasius equation	Adomian decomposition method(ADM) and homotopy perturbation method

			(HPM)
S.Abbasbandy *et al.*	Applied Mathematics and Computation 172 (1) (2006) 431–438	Nonlinear equations	Modified homotopy perturbation method (MHPM) and Adomian decomposition method (ADM)
A. Sadighi and D. D. Ganji	International Journal of Nonlinear Sciences and Numerical Simulation 7 (4) (2006) 411–418 2191-0294	Generalized nonlinear boussinesq equation	Homotopy perturbation method (HPM) and variational iteration method (VIM)
A.M. Wazwaz *et al.*	Kybernetes, 40 (9) (2010) 1305 - 1318	Lane-Emden equations	Adomian decomposition method (ADM) and variational iteration method (VIM)
Sunil Kumar *et al.*	Applied Mathematics and Computation 293 (2016) 508–522	Parabolic singularly perturbed differential equation	Finite element method (FDM), Adomian decomposition method (ADM), Monotone Schwarz iterative method(MSI) and Taylor Galerkin method(TGM)
Z.A.Firoozjae *et al.*	American Journal of Applied Mathematics 3 (3) (2015) 90-94	Nonlinear partial differential equations	Differential quadrature method (DQM) and Adomian decomposition method (ADM)

Table 2: Comparison of various analytical results with numerical results for vibration $x(\tau)$

a=1 b=0.1 Initial value X =0.5							
τ	Numerical simulation	VIM Eqn.(40)	Error(%)	NHPM Eqn.(41)	Error(%)	ADM Eqn.(42)	Error(%)
0	0.5000	0.5000	0.00	0.5000	0.00	0.5000	0.00
0.1	0.4974	0.4975	0.02	0.4974	0.00	0.4974	0.00
0.2	0.4878	0.4898	0.41	0.4896	0.37	0.4896	0.37
0.3	0.4771	0.4772	0.02	0.4766	0.10	0.4767	0.08
0.4	0.4595	0.4597	0.04	0.4587	0.17	0.4588	0.15
0.5	0.4373	0.4376	0.07	0.436	0.30	0.4361	0.27
0.6	0.4106	0.411	0.10	0.4087	0.46	0.409	0.39
0.7	0.3798	0.3803	0.13	0.3771	0.71	0.3776	0.58
0.8	0.3451	0.3458	0.20	0.3416	1.01	0.3425	0.75
0.9	0.3069	0.3079	0.33	0.3024	1.47	0.3039	0.98
1	0.2656	0.2713	2.15	0.2644	0.45	0.2667	0.41
		Average error (%)	0.32	Average error (%)	0.46	Average error (%)	0.36

Table 3: Comparison of various analytical results with numerical results for vibration $x(\tau)$

a=0.01 b=1 Initial value X =0.5							
τ	Numerical simulation	VIM Eqn.(40)	Error(%)	NHPM Eqn.(41)	Error(%)	ADM Eqn.(42)	Error(%)
0	0.5000	0.5000	0.00	0.5000	0.00	0.5000	0.00
0.1	0.4994	0.4993	0.02	0.4993	0.02	0.4993	0.02
0.2	0.4974	0.4974	0.00	0.4973	0.02	0.4974	0.00
0.3	0.4941	0.4941	0.00	0.494	0.02	0.4941	0.00
0.4	0.4897	0.4897	0.00	0.4894	0.06	0.4895	0.04
0.5	0.484	0.4843	0.06	0.4841	0.02	0.4837	0.06
0.6	0.4771	0.4786	0.31	0.4761	0.21	0.4767	0.08
0.7	0.4691	0.4708	0.36	0.4675	0.34	0.4685	0.13
0.8	0.46	0.4631	0.67	0.4576	0.52	0.4593	0.15
0.9	0.4496	0.4559	1.40	0.4463	0.73	0.4502	0.13
1	0.4388	0.4479	2.07	0.4351	0.84	0.4391	0.07
		Average error (%)	0.45	Average error (%)	0.25	Average error (%)	0.06

Table 4: Comparison of various analytical results with numerical results for vibration $x(\tau)$

a=1 b=0.1 Initial value X =0.8							
τ	Numerical simulation	VIM Eqn. (40)	Error(%)	NHPM Eqn. (41)	Error(%)	ADM Eqn. (42)	Error(%)
0	0.8000	0.8000	0.00	0.8000	0.00	0.8000	0.00
0.1	0.7957	0.7959	0.03	0.7959	0.03	0.7959	0.03
0.2	0.783	0.7837	0.09	0.7836	0.08	0.7836	0.08
0.3	0.762	0.7635	0.20	0.7633	0.17	0.7633	0.17
0.4	0.7329	0.7356	0.37	0.7352	0.31	0.7352	0.31
0.5	0.6962	0.7002	0.57	0.6995	0.47	0.6995	0.47
0.6	0.6521	0.6578	0.87	0.6566	0.69	0.6567	0.71
0.7	0.6012	0.6088	1.26	0.607	0.96	0.6073	1.01
0.8	0.5441	0.5538	1.78	0.5511	1.29	0.5517	1.40
0.9	0.4815	0.4895	1.66	0.4896	1.68	0.4907	1.91
1	0.4139	0.4353	5.17	0.4299	3.87	0.4316	4.28
		Average error (%)	1.09	Average error (%)	0.87	Average error (%)	0.88

Table 6: Comparison of various analytical results with numerical results for vibration $x(\tau)$

a=1 b=0.1 Initial value X =1							
τ	Numerical simulation	VIM Eqn. (40)	Error(%)	NHPM Eqn. (41)	Error(%)	ADM Eqn. (42)	Error(%)
0	1000	1000	0.00	1000	0.00	1000	0.00
0.1	0.9945	0.9949	0.04	0.9944	0.01	0.9944	0.01
0.2	0.9781	0.9797	0.16	0.9776	0.05	0.9777	0.04
0.3	0.9509	0.9544	0.37	0.9499	0.11	0.95	0.09
0.4	0.9135	0.9195	0.66	0.9114	0.23	0.9118	0.19
0.5	0.8661	0.8801	1.62	0.8627	0.39	0.8636	0.29
0.6	0.8094	0.828	2.30	0.8041	0.65	0.8122	0.35
0.7	0.7442	0.7677	3.16	0.7435	0.09	0.7469	0.36
0.8	0.671	0.6999	4.31	0.66	1.64	0.6739	0.43
0.9	0.5909	0.6255	5.86	0.5846	1.07	0.5944	0.59
1	0.5047	0.5454	8.06	0.4943	2.06	0.5096	0.97
		Average error (%) 2.41		Average error (%) 0.57		Average error (%) 0.31	

Table 7: Comparison of various analytical results with numerical results for vibration $x(\tau)$

a=0.01 b=1 Initial value X =0.8							
τ	Numerical simulation	VIM Eqn. (40)	Error(%)	NHPM Eqn. (41)	Error(%)	ADM Eqn. (42)	Error(%)
0	0.8000	0.8000	0.00	0.8000	0.00	0.8000	0.00
0.1	0.7974	0.7974	0.00	0.7973	0.01	0.7974	0.00
0.2	0.7896	0.7895	0.01	0.7894	0.03	0.7895	0.01
0.3	0.7769	0.7768	0.01	0.7761	0.10	0.7765	0.05
0.4	0.7594	0.7598	0.05	0.7576	0.24	0.7587	0.09
0.5	0.7375	0.7392	0.23	0.7337	0.52	0.7364	0.15
0.6	0.7115	0.7182	0.94	0.7045	0.98	0.7129	0.20
0.7	0.6818	0.6933	1.69	0.6701	1.72	0.6836	0.26
0.8	0.6488	0.6678	2.93	0.6303	2.85	0.6481	0.11
0.9	0.6131	0.6406	4.49	0.5853	4.53	0.6137	0.10
1	0.575	0.6204	7.90	0.5402	6.05	0.5818	1.18
		Average error (%) 1.66		Average error (%) 1.55		Average error (%) 0.18	

Table 8: Comparison of various analytical results with numerical results for vibration $x(\tau)$

a=1 b=0.01 Initial value X =1							
τ	Numerical simulation	VIM Eqn. (40)	Error(%)	NHPM Eqn. (41)	Error(%)	ADM Eqn. (42)	Error(%)
0	1000	1000	0.00	1000	0.00	1000	0.00
0.1	0.9949	0.995	0.01	0.995	0.01	0.995	0.01
0.2	0.9798	0.9799	0.01	0.9799	0.01	0.9799	0.01
0.3	0.9549	0.9549	0.00	0.9549	0.00	0.9549	0.00
0.4	0.9203	0.9203	0.00	0.9203	0.00	0.9203	0.00
0.5	0.8764	0.8764	0.00	0.8764	0.00	0.8764	0.00
0.6	0.8237	0.8237	0.00	0.8237	0.00	0.8237	0.00
0.7	0.7675	0.7628	0.61	0.7628	0.61	0.7628	0.61
0.8	0.6941	0.6914	0.39	0.6941	0.00	0.6941	0.00
0.9	0.6184	0.6234	0.81	0.6185	0.02	0.6185	0.02
1	0.5366	0.5421	1.02	0.5367	0.02	0.5367	0.02
		Average error (%) 0.26		Average error (%) 0.06		Average error (%) 0.06	

Table 9: Comparison of various analytical results with numerical results for vibration $x(\tau)$

a=0.1 b=1 Initial value X =1							
τ	Numerical simulation	VIM Eqn. (40)	Error(%)	NHPM Eqn. (41)	Error(%)	ADM Eqn. (42)	Error(%)
0	1000	1000	0.00	1000	0.00	1000	0.00
0.1	0.9945	0.9944	0.01	0.9944	0.01	0.9944	0.01
0.2	0.9782	0.9777	0.05	0.9776	0.06	0.778	20.47
0.3	0.9516	0.9503	0.14	0.9495	0.22	0.9507	0.09
0.4	0.9154	0.9128	0.28	0.9103	0.56	0.914	0.15
0.5	0.8707	0.8658	0.56	0.86	1.23	0.869	0.20
0.6	0.8185	0.8161	0.29	0.7986	2.43	0.8171	0.17
0.7	0.76002	0.7466	1.77	0.7262	4.45	0.7664	0.84
0.8	0.6963	0.6833	1.87	0.6516	6.42	0.7074	1.59
0.9	0.6286	0.6067	3.48	0.5486	12.73	0.6245	0.65
1	0.5577	0.5236	6.11	0.4546	18.49	0.5921	6.17
		Average error (%) 1.33		Average error (%) 4.66		Average error (%) 1.31	

Table 10: Comparison of various analytical results with numerical results for vibration $x(\tau)$

a=0.4 b=0.8 Initial value X =1							
τ	Numerical imulation	VIM Eqn. (40)	Error(%)	NHPM Eqn. (41)	Error(%)	ADM Eqn. (42)	Error(%)
0	1000	1000	0.00	1000	0.00	1000	0.00
0.1	0.994	0.9954	0.14	0.9939	0.01	0.9951	0.11
0.2	0.9762	0.9815	0.54	0.9757	0.05	0.9781	0.19
0.3	0.9471	0.9585	1.20	0.9461	0.11	0.9495	0.25
0.4	0.9074	0.9268	2.14	0.9058	0.18	0.9103	0.32
0.5	0.8581	0.8867	3.33	0.8561	0.23	0.8614	0.38
0.6	0.8003	0.8385	4.77	0.7985	0.22	0.8046	0.54
0.7	0.7353	0.7825	6.42	0.735	0.04	0.7416	0.86
0.8	0.6641	0.7191	8.28	0.6747	1.60	0.6679	0.57
0.9	0.5879	0.6483	10.27	0.6065	3.16	0.5998	2.02
1	0.5077	0.5783	13.91	0.54	6.36	0.5465	7.64
		Average error (%) 4.64		Average error (%) 1.09		Average error (%) 1.17	

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