International Journal of Modern Engineering Research (IJMER)
www.ijmer.com Vol. 3, Issue. 3, May.-June. 2013 pp-1487-1492 ISSN: 2249-6645

Observations on Homogeneous Cubic Equation with Four Unknowns

$$X^3 + Y^3 = 7^{2n} ZW^2$$

S. Vidhyalakshmi[1], M. A. Gopalan[2], A. Kavitha[3]

[123]*Professor, PG and Research Department of Mathematics, Shrimati Indira Gandhi college, Trichy-620 002, Tamil nadu, India*

Abstract: *The non-homogeneous cubic equation with three unknowns represented by the diophantine equation $X^3 + Y^3 = 7^{2n} ZW^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers are exhibited.*

Keywords: *Integral solutions, non-homogeneous cubic equation with three unknowns.*

M. Sc 2000 mathematics subject classification: 11D25

Notations:

$t_{m,n}$: Polygonal number of rank n with size m

S_n : Star number of rank n

$\Pr_n$: Pronic number of rank n

j_n : Jacobsthal lucas number of rank n

J_n : Jacobsthal number of rank n

$CP_{m,n}$: Centered Polygonal number of rank n with size m.

G_n : Gnomonic number of rank n

Ky_n : Kynea number of rank n

I. Introduction

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-14] for cubic equations with four unknowns. This communication concerns with yet another interesting equation $X^3 + Y^3 = 7^{2n} ZW^2$ representing non-homogeneous cubic with four unknowns for determining its infinitely many non-zero integral points. Various relations between the solutions and special polygonal numbers, centered polygonal numbers, Jacobsthal numbers and kynea numbers are exhibited.

II. Method of Analysis

The cubic equation with four unknowns to be solved for its distinct non-zero integral solution is

$$x^3 + y^3 = 7^{2n} zw^2 \tag{1}$$

Introduction of the transformations,

$$x = u + v, y = u - v, z = 2u \tag{2}$$

in (1) leads to

$$u^2 + 3v^2 = 7^{2n} w^2 \tag{3}$$

We present below different methods of solving (3) and thus, in view of (2), different patterns of solutions to (1) are obtained

Pattern: 1.1

Let $w = a^2 + 3b^2$ (4)

write 7 as

$$7 = (2 + i\sqrt{3})(2 - i\sqrt{3}) \tag{5}$$

Using (4) and (5) in (3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (2 + i\sqrt{3})^{2n} (a + i\sqrt{3}b)^2 \tag{6}$$

Since the complex number raised to any integer power is also a complex number, we write

$$(2+i\sqrt{3})^{2n} = A_1 + i\sqrt{3}B_1 \tag{7}$$

Where $A_1 = \frac{1}{2}[(2+i\sqrt{3})^{2n} + (2-i\sqrt{3})^{2n}]$

$$B_1 = \frac{1}{2i\sqrt{3}}[(2+i\sqrt{3})^{2n} - (2-i\sqrt{3})^{2n}]$$

Using (7) in (6) and equating the real and imaginary parts, we have

$$\left.\begin{aligned} u &= A_1(a^2-3b^2) - B_1(6ab) \\ v &= A_1(2ab) + B_1(a^2-3b^2) \end{aligned}\right\} \tag{8}$$

Using (8) in (2), we get

$$\left.\begin{aligned} x(a,b) &= A_1(a^2-3b^2+2ab) + B_1(a^2-3b^2-6ab) \\ y(a,b) &= A_1(a^2-3b^2-2ab) - B_1(6ab+a^2-3b^2) \\ z(a,b) &= A_1(2a^2-6b^2) - B_1(12ab) \end{aligned}\right\} \tag{9}$$

Thus, (4) and (9) represent the non-trivial integral solutions of (1)

Properties: 1.2

(i) $x(2^n,1) = A_1(ky_n - 2) + B_1(3J_{2n} - 6j_n + 6(-1)^n - 2)$

(ii) $w(2^n,1) = j_{2n} + 2$

(iii) $z(n,n+1) = -A_1(CP_{8,n} + 16t_{3,n} - 8t_{4,5} + 5) - B_1(24t_{3,n})$

Pattern: 2.1

Write 7 as $7 = \frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4}$ (10)

And $(5+i\sqrt{3})^{2n} = A_2 + i\sqrt{3}B_2$ (11)

Where, $A_2 = \frac{1}{2}[(5+i\sqrt{3})^{2n} + (5-i\sqrt{3})^{2n}]$

$$B_2 = \frac{1}{2i\sqrt{3}}[(5+i\sqrt{3})^{2n} - (5-i\sqrt{3})^{2n}]$$

Using (4), (10), (11) in (3) and employing the method of factorization, we have

$$u + i\sqrt{3}v = \frac{1}{2^{2n}}(A_2 + i\sqrt{3}B_2)(a^2 - 3b^2 + 2i\sqrt{3}ab)$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} u &= \frac{1}{2^{2n}}[A_2(a^2-3b^2) - B_2(6ab)] \\ v &= \frac{1}{2^{2n}}[A_2(2ab) + B_2(a^2-3b^2)] \end{aligned}\right\} \tag{12}$$

Thus, taking $a = 2^n A$ and $b = 2^n B$ the non-zero distinct integral solutions to (1) are given by

$$x(A,B) = A_2(A^2-3B^2+2AB) + B_2(A^2-3B^2-6AB)$$
$$Y(A,B) = A_2(A^2-3B^2-2AB) - B_2(A^2-3B^2+6AB)$$
$$Z(A,B) = A_2(2A^2-6B^2) - B_2(12AB)$$
$$W(A,B) = 2^{2n}(A^2+3B^2)$$

Properties:2.2

$(i) w(2^n,1) = [3J_{2n} + 1][j_{2n} + 2]$

$$(ii)\, y(1,n) = A_2[-2t_{5,n} - 3\Pr_n + 3t_{4,n} + 1] - B_2[-S_n + 3t_{4,n} + 2]$$

$$(iii)\, x(n+1,n) = A_2[4\Pr_n - 4t_{4,n} + 1] - B_2[4t_{4,n} + 4\Pr_n - 1]$$

Pattern:3.1

Introduce the linear transformations

$$u = \alpha + 3T, v = \alpha - T \qquad (13)$$

Let $w = a^2 + 12b^2$ (14)

Write 7 as

$$7 = \frac{(4+i\sqrt{12})(4-i\sqrt{12})}{4} \qquad (15)$$

and $(4+i\sqrt{12})^{2n} = (A_3 + i\sqrt{12}B_3)$ (16)

Where $A_3 = \frac{1}{2}\left[(4+i\sqrt{12}))^{2n} + (4-i\sqrt{12})^{2n}\right]$

$$B_3 = \frac{1}{2i\sqrt{12}}\left[(4+i\sqrt{12}))^{2n} - (4-i\sqrt{12})^{2n}\right]$$

Using (14), (15) and (16) and employing the method of factorization, define

$$2\alpha + i\sqrt{12}T = \frac{1}{2^{2n}}[A_3 + i\sqrt{12}B_3)(a^2 - 12b^2 + i2\sqrt{12}ab)$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} \alpha &= \frac{1}{2^{2n+1}}[A_3(a^2-12b^2) - B_3(24ab)] \\ T &= \frac{1}{2^{2n}}[A_3(2ab) + B_3(a^2-12b^2)] \end{aligned}\right\} \qquad (17)$$

Substituting (17) in (13), we get

$$\left.\begin{aligned} u &= \frac{1}{2^{2n+1}}[A_3(a^2-12b^2+12ab) - B_3(24ab-6a^2+72b^2)] \\ v &= \frac{1}{2^{2n+1}}[A_3(a^2-12b^2-4ab) - B_3(24ab+2a^2-24b^2)] \end{aligned}\right\} \qquad (18)$$

Replacing a by $A2^{n+1}$ and b $B2^{n+1}$, the corresponding integral solutions are given by

$$x(A,B) = 2[A_3(2A^2 - 24B^2 + 8AB) - B_3(48AB - 4A^2 + 48B^2)]$$

$$y(A,B) = 2[A_3(16AB) - B_3(-8A^2 + 96B^2)]$$

$$z(A,B) = 4[A_3(A^2 - 12B^2 + 12AB) - B_3(24AB - 6A^2 + 72B^2)]$$

$$w(A,B) = 2^{2n+2}(A^2 + 12B^2)$$

Properties: 3.2

$$(i)\, x(n,1) = 2[A_3(CP_{4,n} + 6\Pr_n - 6t_{4,n} - 25) - B_3(-t_{10.n} + 45\Pr_n - 45t_{4,n} + 48)]$$

$$(ii)\, y(n+1,n) = A_3(32\Pr_n) - 16B_3(t_{24,n} + 8\Pr_n - 8t_{4,n} + 1)$$

$$(iii)\, w(2^n,1) = j_{4n+1} + 9J_{2n+3} - 2$$

Note: .3.3

Replacing (13) by $u = \alpha - 3T$ and $v = \alpha + T$ (19)

And repeating the process as in pattern.3 the corresponding non-zero distinct integral solutions to (1) are obtain as

$$x(A,B) = 2[A_3(2A^2 - 24B^2 - 8AB) + B_3(-48AB + A^2 - 12B^2)]$$

$$y(A,B) = 2[A_3(-16AB) - B_3(3A^2 - 36B^2)]$$

$$z(A,B) = 4[A_3(A^2 - 12B^2 - 12AB) - B_3(24AB + A^2 - 12B^2)]$$

$$w(A,B) = 2^{2n+2}(A^2 + 12B^2)$$

Properties: 3.4

$$(i)\, x(n,1) = A_3(t_{10,n} - 13\Pr_n + 13t_{4,n} - 48) + B_3(t_{6,n} - 95\Pr_n + 95t_{4,n} - 24)$$

$$(ii)\, y(n,1) = A_3(-32\Pr_n + 32t_{4,n}) + 6B_3(12t_{4,n} - 1)$$

$$(iii)\, z(n,1) = A_3(t_{10,n} - 45\Pr_n + 45t_{4,n} - 48) - B_3(CP_{8,n} + 95\Pr_n - 95t_{4,n} - 49)$$

Pattern:4.1

Instead of (15) we write 7 as

$$7 = \frac{(10 + i\sqrt{12})(10 - i\sqrt{12})}{16} \qquad (20)$$

And $(10 + i\sqrt{12})^{2n} = (A_4 + i\sqrt{12}B_4)$ (21)

Where $A_4 = \frac{1}{2}[(10 + i\sqrt{12})^{2n} + (10 - i\sqrt{12})^{2n}$

$$B_4 = \frac{1}{2i\sqrt{12}}[(10 + i\sqrt{12})^{2n} - (10 - i\sqrt{12})^{2n}$$

Using (14), (20) and (21) and equating real and imaginary parts, we have

$$\left.\begin{aligned} \alpha &= \frac{1}{2^{4n+1}}[A_4(a^2 - 12b^2) - B_4(24ab)] \\ T &= \frac{1}{2^{4n}}[A_4(2ab) + B_4(a^2 - 12b^2)] \end{aligned}\right\} \qquad (22)$$

Substituting (22) in (13), we get

$$u = \frac{1}{2^{4n+1}}[A_4(a^2 - 12b^2 + 12ab) - B_4(24ab - 6a^2 + 72b^2)]$$

$$v = \frac{1}{2^{4n+1}}[A_4(a^2 - 12b^2 - 4ab) - B_4(24ab + 2a^2 - 24b^2)]$$

To get a integer solution replacing a by $2^{n+1}A$ and b by $2^{n+1}B$

$$x(A,B) = 2[A_4(2A^2 - 24B^2 + 8AB) - B_4(48AB - 4A^2 + 48B^2)]$$

$$y(A,B) = 2[A_4(16AB) - B_4(-8A^2 + 96B^2)]$$

$$z(A,B) = 4[A_4(A^2 - 12B^2 + 12AB) - B_4(24AB - 6A^2 + 72B^2)]$$

$$w(A,B) = 2^{4n+2}(A^2 + 12B^2)$$

Properties: 4.2

(i) $x(2^n,1) = 4A_4(ky_n + 2j_n - 11 - 2(-1)^n) + 8B_4(ky_n - 14j_n - 11 + 14(-1)^n)$

(ii) $y(n,1) = A_4(32t_{3,n} - 32t_{4,n}) - B_4(-16t_{4,n} + 192)$

(iii) $z(n+1,n) = 4[A_4(t_{4,n} + 7G_n + 8) - B_4(CP_{16,n} + CP_{20,n} + 78t_{4,n} - 8)]$

Note: 4.3

Using (19) and repeating the process as in pattern.4, the non-zero distinct integral solutions to (1) are given by

$$x(A,B) = 2[A_4(2A^2 - 24B^2 - 8AB) - B_4(48AB + 4A^2 - 48B^2)]$$

$$y(A,B) = 2[A_4(-16AB) - B_4(8A^2 - 96B^2)]$$

$$z(A,B) = 4[A_4(A^2 - 12B^2 - 12AB) - B_4(24AB + 6A^2 - 72B^2)]$$

$$w(A,B) = 2^{4n+2}(A^2 + 12B^2)$$

Properties:4.4

$(i)x(n,1)=A_4[4(t_{6,n}-3\Pr_n+2t_{4,n}-12)]-8B_4[CP_{4,n}+10\Pr_n-11t_{4,n}-13]$

$(ii)y(n+1,n)=-64A_4t_{3,n}+16B_4(t_{8,n}+8t_{4,n}-1)$

$(iii)z(n+1,n)=-4A(CP_{20,n}+13t_{4,n}-1)+24B_4(t_{12,n}+2t_{4,n})$

Pattern: 5.1

(3) can be written as

$$\frac{3v}{7^n w-u}=\frac{7^n w+u}{v}=\frac{p}{q},q\neq 0 \qquad (23)$$

Which is equivalent to the system of equations

$$pu+3vq-7^n wp=0 \qquad (24)$$

$$qu-pv+7^n qw=0 \qquad (25)$$

Applying the cross-multiplication method, we get

$u=7^n(3q^2-p^2)$

$v=-2*7^n pq$

$w=-p^2-3q^2$

Thus, the corresponding non zero distinct integral solutions to (1) are given by

$x=7^n(3q^2-p^2-2pq)$

$y=7^n(3q^2-p^2+2pq)$

$z=2*7^n(3q^2-p^2)$

$w=-p^2-3q^2$

Properties: 5.2

$(i)7^n[x(7^n,1)]$ is a difference of two square

$(ii)x(7^n,1)+y(7^n,1)\equiv 0(\text{mod}\,7)$

$(iii)6*7^n[x(7^n,7^n)-y(7^n,7^n)]$ is a nasty number

$(iv)z(7^n,7^n)$ is a perfect square

$(v)w(2^n,1)=-(j_{2n}+2)$

Pattern: 5.3

(23) can be written as

$$\frac{v}{7^n w-u}=\frac{7^n w+u}{3v}=\frac{p}{q} \qquad (26)$$

Repeating the process as in pattern.5, the non-zero distinct integral solutions to (1) are obtain as

$x=7^n(q^2-3p^2-2pq),\qquad y=7^n(q^2-3p^2+2pq)$

$z=2*7^n(q^2-3p^2),\qquad w=-(3p^2+q^2)$

Properties: 5.4

$(i)x(n,1)=-7^n(CP_{6,n}-\Pr_n+t_{4,n}-2)$

$(ii)y(1,n)=7(CP_{4,n}-t_{4,n}-4)$

$(iii)z(n,n+1)=2*7(-2t_{4,n}+G_n+2)$

III. Conclusion

To conclude, one may search for other pattern of solutions and their corresponding properties

References

[1] Dickson. I. E. ‘”History of the Theory of Numbers’’ (Vol 2. Diophantine analysis, New York, Dover, 2005).
[2] Mordell .L J., “Diophantine Equations’’ (Academic Press, New York, 1969)
[3] Carmichael. R.D. “The Theory of numbers and Diophantine Analysis’’, (New York, Dover, 1959).
[4] M.AGopalan and K.Geetha, Observations on cubic equation with four unknowns $x^3+y^3+xy(x+y)=z^3+2(x+y)w^2$, International journal of pure and Applied Mathematical sciences, Vol.6, No: 1, 2013, 25-30
[5] M.A.Gopalan and S.Premalatha, Integral solutions of $(x+y)(xy+w^2)=2(k^2+1)z^3$, Bulletin of Pure and Applied sciences, vol.29E,No:2 2009,197-202.
[6] M.A.Gopalan and V.Pandichelvi, Remarkable solutions on the cubic equations with four unknowns $x^3+y^3+z^3=28(x+y+z)w^2$, Antarctica J.Math., Vol.7, No.4, 2010, 393-401,
[7] M.A.Gopalan and S.Premalatha, On the cubic Diophantine equation with 4 unknowns $(x-y)(xy-w^2)=2(n^2=2n)z^3$, Integral journal of Mathematical sciences, Vol.9, No.12, Jan-June 2010, 171-175
[8] M.A.Gopalan and J.Kaligarani, Integral solutions of $x^3+y^3+(x+y)xy=z^3+w^3+zw(z+w)$, Bulletin of Pure and Applied sciences, Vol.29E, No.1, 2010, 169-173
[9] M.A.Gopalan and S.Premalatha, Integral solutions of $(x+y)(xy+w^2)=2(k+1)z^3$, the global journal of Applied Mathematics and mathematical sciences, Vol.3, N0:1-2, Jan-Dec.2010, 51-55
[10] M.A.Gopalan, S.Vidhyalakshmi and S.Mallika, Observation on the Cubic equations with 4 unknowns $xy+2z^2=w^3$, The Global journal of Mathematics and Mathematical Sciences, Vol.2, No.1, 2012,69-74
[11] M.A.Gopalan V.Sangeetha and Manju Somanath, Lattice points on the homogeneous cubic equation with 4 unknowns $(x+y)(xy+w^2)=(k^2-1)z^3, k>1$, Indian journal of sciences, vol.2, No.4, 2013, 97-99.
[12] M.A.Gopalan, S.Vidhyalakshmi and G.Sumathi On the homogeneous cubic equation with 4 unknowns $x^3+y^3=14z^3-3w^2(x+y)$, Discovery, Vol.2, No.4, Oct.2012, 17-19
[13] M.AGopalan, S. Vidhyalakshmi and S.Mallika,, Observation on the Cubic equations with 4 unknowns $2(x^3+y^3)=z^3+w^2(x+y)$, IJAMP, Vol.4,No.2, July-Dec. 2012, 103-107,
[14] M.A.Gopalan and V.Pandichelvi, on the cubic equations with four unknowns $x^2-xy+y^2+k^2+2kw=(k^2+3)z^3$, Impact J.Sci.Tech., Vol.1, No.1 2012, 81-86