International Journal of Modern Engineering Research (IJMER)
www.ijmer.com Vol.3, Issue.3, May-June. 2013 pp-1424-1427 ISSN: 2249-6645

Integral solution of the non-homogeneous heptic equation in terms of the generalised Fibonacci and Lucas sequences

$$x^5+y^5-(x^3+y^3)xy-4z^2w=3(p^2-T^2)^2w^3$$

S.Vidhyalakshmi, [1] K.Lakshmi, [2] M.A.Gopalan[3]

[123]*Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002*

Abstract: *We obtain infinitely many non-zero integer sextuples* (x,y,z,w,p,T) *satisfying the Non-homogeneous equation of degree seven with six unknowns given by* $x^5+y^5-(x^3+y^3)xy-4z^2w=3(p^2-T^2)^2w^3$. *The solutions are obtained in terms of the generalised Fibonacci and Lucas sequences. Recurrence relations for the variables are given. Various interesting relations between the solutions and special numbers, namely polygonal numbers, Pyramidal numbers, centered hexagonal pyramidal numbers and Four Dimensional Figurative numbers are presented.*

Keywords: *Fibonacci and Lucas sequences, heptic equation, integral solutions, Non-homogeneous equation, special numbers.*

MSC 2000 Mathematics subject classification: 11D41.

NOTATIONS:

$$GF_n(k,s)=\frac{\alpha^n-\beta^n}{\alpha-\beta}\left(\alpha=\frac{k+\sqrt{k^2+4s}}{2},\beta=\frac{k-\sqrt{k^2+4s}}{2}\right)$$ -Generalised Fibonacci sequence

$$GL_n(k,s)=\alpha^n+\beta^n\left(\alpha=\frac{k+\sqrt{k^2+4s}}{2},\beta=\frac{k-\sqrt{k^2+4s}}{2}\right)$$ -Generalised Lucas sequence

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$F_{4,n,3}$ -Four Dimensional Figurative number of rank n whose generating polygon is a triangle

I. Introduction

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1-3].Particularly in [4, 5] special equations of sixth degree with four and five unknowns are studied. In [6-8] heptic equations with three and five unknowns are analysed. This paper concerns with the problem of determining non-trivial integral solution of the non- homogeneous equation of seventh degree with six unknowns given by $x^5+y^5-(x^3+y^3)xy-4z^2w=3(p^2-T^2)^2w^3$, in terms of the generalised fibonacci and lucas sequences. Recurrence relations for the variables are also given.Various interesting properties between the solutions and special numbers are presented.

II. Method of Analysis

The Diophantine equation representing the non- homogeneous equation of degree seven is given by

$$x^5+y^5-(x^3+y^3)xy-4z^2w=3(p^2-T^2)^2w^3 \tag{1}$$

Introduction of the transformations

$$x=u+v, y=u-v, z=2v, w=u, p=v+1, T=v-1,\ \ v>1 \tag{2}$$

In (1) leads to $v^2-2u^2=1$ (3)

The above equation (3) is a pellian equation, whose general solution is given by

$$\left.\begin{aligned} v_n &= \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n+1}+\left(3-2\sqrt{2}\right)^{n+1}\right] \\ u_n &= \frac{1}{2\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{n+1}-\left(3-2\sqrt{2}\right)^{n+1}\right] n=0,1,2,\ldots\ldots \end{aligned}\right\} \quad (4)$$

The values of u_n and v_n can be written in terms of the generalised Fibonacci and Lucas sequences.

$$\left.\begin{aligned} v_n &= \frac{1}{2}GL_{n+1}(6,-1) \\ u_n &= 2GF_{n+1}(6,-1) \end{aligned}\right\} \quad (5)$$

In view of (2) and (5) the non-zero distinct integral solutions of (1) in terms of the generalised Fibonacci and Lucas sequences are obtained as

$$\left.\begin{aligned} x_n &= 2GF_{n+1}(6,-1)+\frac{1}{2}GL_{n+1}(6,-1) \\ y_n &= 2GF_{n+1}(6,-1)-\frac{1}{2}GL_{n+1}(6,-1) \\ z_n &= GL_{n+1}(6,-1) \\ w_n &= 2GF_{n+1}(6,-1) \\ p_n &= \frac{1}{2}GL_{n+1}(6,-1)+1 \\ T_n &= \frac{1}{2}GL_{n+1}(6,-1)-1, \quad n=0,1,2,3.... \end{aligned}\right\} \quad (6)$$

A few numerical examples are tabulated below:

n	x_n	y_n	z_n	w_n	p_n	T_n
0	5	-1	6	2	4	2
1	29	-5	34	12	18	16
2	169	-29	198	70	100	98
3	985	-169	1154	408	578	576
4	5741	-985	6726	2378	3364	3362
5	33461	-5741	39202	13860	19602	19600
6	195025	-33461	228486	80782	114244	114242

The above values of x_n, y_n, z_n, w_n, p_n, T_n, satisfy the following recurrence relations respectively.

$$x_{n+2}-6x_{n+1}+x_n=0$$
$$y_{n+2}-6y_{n+1}+y_n=0$$
$$z_{n+2}-6z_{n+1}+z_n=0$$
$$w_{n+2}-6w_{n+1}+w_n=0$$
$$p_{n+2}-6p_{n+1}+p_n=-4$$
$$T_{n+2}-6T_{n+1}+T_n=4$$

2.1 Properties:

1. $2GF_{n+1}(6,-1)+\frac{1}{2}GL_{n+1}(6,-1)+2GF_{n+2}(6,-1)-\frac{1}{2}GL_{n+2}(6,-1)=0$
2. $x_n+y_n+2w_n+z_n+p_n+T_n=8GF_{n+1}(6,-1)+2GL_{n+1}(6,-1)$
3. $x_n-y_n=p_n+T_n$
4. $T_{2n+1}-\frac{1}{2}GL_{2n+2}(6,-1)+1=0$

5. $x_{3n+2} - y_{3n+2} = GL_{3n+3}(6,-1)$
6. $w_{2n+1} = 2GL_{n+1}(6,-1).GF_{n+1}(6,-1)$
7. $GL_{3n+3}(6,-1) - 2T_{3n+2} = 0(\text{mod } 2)$
8. $x_n + y_n = 2w_n$
9. Each of the following is a nasty number:
a) $6(z_{2n+1} + 4)$
b) $3(2p_{2n+1} - GL_{n+1}(6,-1))$
c) $6(x_{2n+1} - y_{2n+1}) + 2p_2^5$
d) $48w_n^2 + 24F_{4,1,3}$
10. Each of the following is a cubical integer:
a) $4(z_n^2 + 2z_n w_n - 2x_{2n+1})$
b) $2p_{3n+2} + 3z_n - 2$
c) $x_{3n+2} - y_{3n+2} + 3z_n$
d) $x_{3n+2} - y_{3n+2} - 6p_n - 6$
e) $2T_{3n+2} + 3(x_n - y_n) + 2$
f) $2T_{3n+2} + 3z_n + 2$
g) $2T_{3n+2} + 6p_n - 4$
11. Each of the following is a biquadratic integer:
a) $2p_{4n+3} + 8T_{2n+1} + 2p_2^5$
b) $x_{4n+3} - y_{4n+3} + 4z_{2n+1} + 2T_{3,2}$
c) $2T_{4n+3} + 4(x_{2n+1} - y_{2n+1}) + CP_{2,6}$
d) $2T_{4n+3} + 8T_{2n+1} + T_{4,4}$
e) $8(z_{2n+1} - 8w_n^2)$
g) $2p_{4n+3} + 4(x_{2n+1} - y_{2n+1} + 2)$
h) $x_{4n+3} - y_{4n+3} + 4(x_{2n+1} - y_{2n+1}) + 6$

12. $2z_n w_n - z_{2n+1} - 2y_{2n+1} = 0$
13. $w_{2n+1} - z_n w_n = 0$
14. $x_{2n+1} + y_{2n+1} - 2z_n w_n = 0$
15. $x_{2n+1} + y_{2n+1} - 4p_n w_n + 4w_n = 0$
16. $z_n^2 - x_{2n+1} + y_{2n+1} \equiv 0(\text{mod } 2)$
17. $x_{3n+2} + y_{3n+2} = 2w_n(z_{2n+1} + 1)$
18. $w_{2n+1} - 2w_n p_n + 2w_n = 0$

19. Define:

$X = z_n,\ \mathrm{Y}=2(\mathrm{p_n} - 1),\ Z = z_{2n+1} + 2$

It is to be noted that the triple (X, Y, Z) satisfies the Elliptic Paraboloid $X^2 + Y^2 = 2Z$

20. Define:

$X = z_n,\ \mathrm{Y}=2\mathrm{T_n} + 2,\ \mathrm{w} = \mathrm{x}_{2n+1} - y_{2n+1} + 2$

It is to be noted that the triple (X, Y, W) satisfies the Hyperbolic Paraboloid $2X^2 - Y^2 = W$

21. Define:

$X = z_{2n+1} + 2,\ \mathrm{Y}=2\mathrm{T}_{2n+1} + 4,\ \mathrm{Z}=\ \mathrm{x}_{2n+1} - y_{2n+1} + 2$

It is to be noted that the triple (X, Y, Z) satisfies the Cone $X^2 + Y^2 = 2Z^2$

22. Define:

$X = 2T_n + 2,\ \mathrm{Y}=\ \mathrm{x}_{2n+1} - y_{2n+1} + 2$

It is to be noted that the pair (X, Y) satisfies the parabola $X^2 = Y$

III. Conclusion

One may be able to get the solutions to (1) in terms of other choices of number sequences. For example, the solution to (1) is also written in terms of Pell and Pell-Lucas sequence as follows:

$$\left.\begin{aligned}
x_n &= 2P_{2n+2}(2,1) + \frac{1}{2} PL_{2n+2}(2,1) \\
y_n &= 2P_{2n+2}(2,1) - \frac{1}{2} PL_{2n+2}(2,1) \\
z_n &= PL_{2n+2}(2,1) \\
w_n &= 2P_{2n+2}(2,1) \\
p_n &= \frac{1}{2} PL_{2n+2}(2,1) + 1 \\
T_n &= \frac{1}{2} PL_{2n+2}(2,1) - 1,\ n = 0,1,2,3....
\end{aligned}\right\} \quad (7)$$

The corresponding properties can also be obtained in terms of number sequences.

References

[1] L.E.Dickson, History of Theory of Numbers Vol.11, Chelsea Publishing company,New York 1952.

[2] L.J.Mordell, Diophantine equation Academic Press, London1969

[3] Carmichael, R.D.,The theory of numbers and Diophantine Analysis Dover Publications, New York 1959

[4] M.A.Gopalan, S.Vidhyalakshmi and K.Lakshmi, on the non- homogeneous sextic equation $x^4 + 2(x^2 + w)x^2y^2 + y^4 = z^4$, International Journal of Applied Mathematics and Applications, 4(2), 2012, 171-173

[5] M.A.Gopalan, S.Vidhyalakshmi and K.Lakshmi, Integral Solutions of the sextic equation with five Unknowns $x^3 + y^3 = z^3 + w^3 + 3(x+y)T^5$ International Journal of Engineering Sciences & Research Technology, 1(10,) 2012, 562-564

[6] M.A.Gopalan and sangeetha.G, parametric integral solutions of the heptic equation with 5unknown $x^4 - y^4 + 2(x^3 + y^3)(x-y) = 2(X^2 - Y^2)z^5$, Bessel Journal of Mathematics 1(1), 2011, 17-22

[7] M.A.Gopalan and sangeetha.G, on the heptic diophantine equations with 5 unknowns $x^4 - y^4 = (X^2 - Y^2)z^5$, Antarctica Journal of Mathematics, 9(5), 2012, 371-375

[8] Manjusomnath, G.sangeetha and M.A.Gopalan, on the non-homogeneous heptic equations with 3 unknowns $x^3 + (2^p - 1)y^5 = z^7$, Diophantine journal of Mathematics, 1 (2), 2012, 17-121