The (G'/G) - Expansion Method for Finding Traveling Wave Solutions of Some Nonlinear Pdes in Mathematical Physics

J.F.Alzaidy
Mathematics Department, Faculty of Science, Taif University, Kingdom of Saudi Arabia

Abstract: In the present paper, we construct the traveling wave solutions involving parameters of some nonlinear PDEs in mathematical physics; namely the variable coefficients KdV (vcKdV) equation, the modified dispersive water wave (MDWW) equations and the symmetrically coupled KdV equations by using a simple method which is called the $\left(\frac{G'}{G}\right)$-expansion method, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation. When the parameters are taken special values, the solitary waves are derived from the traveling waves. The traveling wave solutions are expressed by hyperbolic, trigonometric and rational functions. This method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear PDEs arising in mathematical physics.

Keywords: *The(G'/G) expansion method; traveling wave solutions; the variable coefficients KdV (vcKdV) equation; the modified dispersive water wave (MDWW) equations; symmetrically coupled KdV equations, solitary wave solutions.*

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I. Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors (see for example [1-30]) who are interested in nonlinear physical phenomena. Many effective methods have been presented ,such as inverse scattering transform method [1], Bäcklund transformation [2], Darboux transformation [3], Hirota bilinear method [4], variable separation approach [5], various tanh methods [6–9], homogeneous balance method [10] , similarity reductions method [11,12] , the reduction mKdV equation method [13], the tri-function method [14,15], the projective Riccati equation method [16], the Weierstrass elliptic function method [17], the Sine- Cosine method [18,19], the Jacobi elliptic function expansion [20,21], the complex hyperbolic function method [22], the truncated Painleve' expansion [23], the F-expansion method [24], the rank analysis method [25] and so on.

In the present paper, we shall use a simple method which is called the $\left(\frac{G'}{G}\right)$-expansion method [26,27]. This method is firstly proposed by the Chinese Mathematicians Wang et al [28] for which the traveling wave solutions of nonlinear equations are obtained. The main idea of this method is that the traveling wave solutions of nonlinear equations can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = k(x - ct)$,where λ, μ, k and c are constants . The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations .The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method. This new method will play an important role in expressing the traveling wave solutions for nonlinear evolution equations via the vcKdV equation, the MDWW equations and the symmetrically coupled KdV equations in terms of hyperbolic, trigonometric and rational functions.

II. Description of the $\left(\frac{G'}{G}\right)$- expansion method

Suppose we have the following nonlinear PDE:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{1}$$

Where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of a deformation method:

Setp1. Suppose that

$$u(x,t) = u(\xi), \quad \xi = \xi(x,t). \tag{2}$$

The traveling wave variable (2) permits us reducing (1) to an ODE for $u = u(\xi)$ in the form:

$$P(u, u', u'', \ldots) = 0, \tag{3}$$

where $' = \frac{d}{d\xi}$.

Setp2. Suppose that the solution Eq.(3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i, \tag{4}$$

While $G = G(\xi)$ satisfies the second order linear differential equation in the form:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{5}$$

Where $a_i\,(i = 0,1,\dots,n), \lambda$ and μ are constants to be determined later.

Setp3. The positive integer " n " can be determined by considering the homogeneous balance between the highest derivative term and the nonlinear terms appearing in Eq.(3). Therefore, we can get the value of n in Eq.(4).

Setp4. Substituting Eq.(4) into Eq.(3) and using Eq.(5), collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together and then equating each coefficient of the resulted polynomial to zero, yield a set of algebraic equations for a_i, λ, μ, c and k.

Setp5. Solving the algebraic equations by use of Maple or Mathematica, we obtain values for a_i, λ, μ, c and k.

Setp6. Since the general solutions of Eq. (5) have been well known for us, then substituting the obtained coefficients and the general solution of Eq. (5) into Eq. (4), we have the travelling wave solutions of the nonlinear PDE (1).

III. Applications of the method

In this section, we apply the $\left(\frac{G'}{G}\right)$-expansion method to construct the traveling wave solutions for some nonlinear PDEs, namely the vcKdV equation, the MDWW equations and the symmetrically coupled KdV equations which are very important nonlinear evolution equations in mathematical physics and have been paid attention by many researchers.

3.1. Example1. The cvKdV equation

We start with the cvKdV equation [29] in the form:

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0, \tag{6}$$

Where $f(t) \neq 0, g(t) \neq 0$ are some given functions.This equation is well-known as a model equation describing the propagation of weakly-nonlinear weakly-dispersive waves in inhomogeneous media.. Obtaining exact solutions for non-linear differential equations have long been one of the central themes of perpetual interest in mathematics and physics. To study the travelling wave solutions of Eq. (6), we take the following transformation

$$u(x,t) = u(\xi), \qquad \xi = x + \frac{\omega}{\alpha}\int_0^t g(\hat{t})\,d\,\hat{t}, \tag{7}$$

Where $\omega$the wave is speed and α is a constant. By using Eq. (7), Eq.(6) is converted into an ODE

$$\frac{\omega}{\alpha}u' + 2uu' + u''' = 0, \tag{8}$$

Where the functions $f(t)$ and $g(t)$ in Eq.(6) should satisfy the condition

$$f(t) = 2g(t). \tag{9}$$

Integrating Eq.(8) with respect to ξ once and taking the constant of integration to be zero, we obtain

$$\frac{\omega}{\alpha}u + u^2 + u'' = 0, \tag{10}$$

Suppose that the solution of ODE (10) can be expressed by polynomial in terms of $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i, \tag{11}$$

Where $a_i\,(i = 0,1,\dots,n)$ are arbitrary constants, while $G(\xi)$ satisfies the second order linear ODE (5). Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (10), we get $n = 2$. Thus, we have

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \tag{12}$$

Where a_0, a_1 and a_2 are constants to be determined later. Substituting Eq.(12) with Eq.(5) into Eq.(10) and collect-ing all terms with the same power of $\left(\frac{G'}{G}\right)$. Setting each coefficients of this polynomial to be zero, we have the following system of algebraic equations:

$\left(\frac{G'}{G}\right)^0$: $\frac{\omega a_0}{\alpha} + a_0^2 + \lambda\mu a_1 + 2\mu^2 a_2 = 0,$

$\left(\frac{G'}{G}\right)^1$: $\lambda^2 a_1 + 2\mu a_1 + \frac{\omega a_1}{\alpha} + 2a_0 a_1 + 6\lambda\mu a_2 = 0,$

$\left(\frac{G'}{G}\right)^2$: $3\lambda a_1 + a_1^2 + 4\lambda^2 a_2 + 8\mu a_2 + \frac{\omega a_2}{\alpha} + 2a_0 a_2 = 0,$

$\left(\frac{G'}{G}\right)^3$: $2a_1 + 10\lambda a_2 + 2a_1 a_2 = 0,$

$$\left(\frac{G'}{G}\right)^4 : \ 6a_2 + a_2^2 = 0, \tag{13}$$

On solving the above algebraic Eqs. (13) By using the Maple or Mathematica, we have

$$a_0 = -6\mu, \qquad a_1 = -6\lambda, \qquad a_2 = -6, \qquad \omega = -\alpha M, \tag{14}$$

Where $M = \lambda^2 - 4\mu$.

Substituting Eq. (14) into Eq.(12) yields

$$u(\xi) = -6\mu - 6\lambda\left(\frac{G'}{G}\right) - 6\left(\frac{G'}{G}\right)^2, \tag{15}$$

Where

$$\xi = x - M\int_0^t g(\hat{t})d\,\hat{t}. \tag{16}$$

On solving Eq.(5), we deduce that

$$\frac{G'}{G} = \begin{cases} \frac{1}{2}\sqrt{M}\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}\right) - \frac{\lambda}{2} & \text{if} \quad M > 0, \\ \frac{1}{2}\sqrt{-M}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{-M}\,\xi\right) + B\cos\left(\frac{1}{2}\sqrt{-M}\,\xi\right)}{A\cos\left(\frac{1}{2}\sqrt{-M}\,\xi\right) + B\sin\left(\frac{1}{2}\sqrt{-M}\,\xi\right)}\right) - \frac{\lambda}{2} & \text{if} \quad M < 0, \\ \frac{B}{B\xi + \mathrm{A}} - \frac{\lambda}{2} & \text{if} \quad M = 0, \end{cases} \tag{17}$$

Where A and B are arbitrary constants and $M = \lambda^2 - 4\mu$.
On substituting Eq.(17) into Eq.(15), we deduce the following three types of traveling wave solutions:

Case1. If $M > 0$, Then we have the hyperbolic solution

$$u(\xi) = \frac{3M}{2}\left[1 - \left(\frac{A\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}\right)^2\right], \tag{18}$$

Case2. If $M < 0$, Then we have the trigonometric solution

$$u(\xi) = \frac{3M}{2}\left[1 + \left(\frac{-A\sin\left(\frac{1}{2}\sqrt{-M}\,\xi\right) + B\cos\left(\frac{1}{2}\sqrt{-M}\,\xi\right)}{A\cos\left(\frac{1}{2}\sqrt{-M}\,\xi\right) + B\sin\left(\frac{1}{2}\sqrt{-M}\,\xi\right)}\right)^2\right], \tag{19}$$

Case3. If $M = 0$, Then we have the the rational solution

$$u(\xi) = \frac{3}{2}\lambda^2 - 6\left[\mu + \left(\frac{B}{B\xi + \mathrm{A}}\right)^2\right], \tag{20}$$

In particular, if we set $B = 0,\ A \neq 0,\ \lambda > 0,\ \mu = 0,$ in Eq.(18), then we get

$$u(\xi) = -\frac{3\lambda^2}{2}\operatorname{csch}^2\left(\frac{\lambda}{2}\xi\right), \tag{21}$$

While, if $B \neq 0$, $\lambda > 0$, $A^2 > B^2$ *and* $\mu = 0$, then we deduce that:

$$u(\xi) = \frac{3\lambda^2}{2}\operatorname{sech}^2\left(\frac{\lambda}{2}\xi + +\xi_0\right), \tag{22}$$

Where $\xi_0 = \tanh^{-1}\left(\frac{A}{B}\right)$.The solutions (21) and (22) represent the solitary wave solutions of Eq. (6).

3.2. Example 2. The MDWW equation

In this subsection, we study the MDWW equations [30] in the forms:

$$u_t = -\frac{1}{4}v_{xx} + \frac{1}{2}(uv)_x\,, \tag{23}$$

$$v_t = -u_{xx} - 2uu_x + \frac{3}{2}vv_x. \tag{24}$$

The traveling wave variables below

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad \xi = k(x + \omega t), \tag{25}$$

permit us converting the equations (23) and (24) into ODEs for $u(x,t) = u(\xi)$ and $v(x,t) = v(\xi)$ as follows:

$$-\frac{1}{4}kv'' + \frac{1}{2}(uv)' - \omega u' = 0, \tag{26}$$

$$-ku'' - 2uu' + \frac{3}{2}vv' - \omega v' = 0, \tag{27}$$

Where k and ω are the wave number and the wave speed, respectively. On integrating Eqs.(26) and (27) with respect to ξ once, yields

$$\mathrm{k}_1 - \frac{1}{4}kv' + \frac{1}{2}(uv) - \omega u = 0, \tag{28}$$

$$\mathrm{k}_2 - ku' - u^2 + \frac{3}{4}v^2 - \omega v = 0, \tag{29}$$

Where k_1 and k_2 is an integration constants.

Suppose that the solutions of the ODEs (28) and (29) can be expressed by polynomials in terms of $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i\left(\frac{G'}{G}\right)^i, \tag{30}$$

$$v(\xi) = \sum_{i=0}^{m} b_i\left(\frac{G'}{G}\right)^i. \tag{31}$$

Where $a_i(i = 0,1,\dots,n)$ and $b_i(i = 0,1,\dots,m)$ are arbitrary constants, while $G(\xi)$ satisfies the second order linear ODE (5).Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eqs.(28) and (29), we get $n = m = 1$. Thus, we have

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0, \tag{32}$$

$$v(\xi) = b_0 + b_1\left(\frac{G'}{G}\right), \quad b_1 \neq 0, \tag{33}$$

Where a_0, a_1, b_0 and b_1 are arbitrary constants to be determined later. Substituting Eqs.(32),(33) with Eq.(5) into Eqs.(28) and (29), collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ and setting them to zero, we have the following system of algebraic equations:

$$\left(\frac{G'}{G}\right)^0: \quad \omega a_0 + \frac{a_0 b_0}{2} + \frac{1}{4}k\mu b_1 + k_1 = 0,$$

$$\left(\frac{G'}{G}\right)^1: \omega a_1 + \frac{a_1 b_0}{2} + \frac{1}{4}k\lambda b_1 + \frac{a_0 b_1}{2} = 0,$$

$$\left(\frac{G'}{G}\right)^2: \frac{kb_1}{4} + \frac{a_1 b_1}{2} = 0,$$

$$\left(\frac{G'}{G}\right)^0 : \; -a_0^2 + k\mu a_1 + \omega b_0 + \frac{3b_0^2}{4} + k_2 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : \; k\lambda a_1 - 2a_0 a_1 + \omega b_1 + \frac{3b_0 b_1}{2} = 0,$$

$$\left(\frac{G'}{G}\right)^2 : \; k a_1 - a_1^2 + \frac{3b_1^2}{4} = 0. \tag{34}$$

Solving the above algebraic Eqs.(34) by using the Maple or Mathematica, yields

$$a_0 = \frac{1}{2}(\lambda a_1 - \omega), \qquad a_1 = a_1, \qquad b_o = \lambda a_1 - \omega, \qquad b_1 = 2a_1,$$
$$k_1 = \frac{1}{4}(\omega^2 - M a_1^2), \qquad k_1 = \frac{1}{2}(\omega^2 - M a_1^2), \qquad k = -2a_1. \tag{35}$$

Substituting Eq.(35) into Eqs.(32) and (33) we obtain

$$u(\xi) = \frac{1}{2}(\lambda a_1 - \omega) + a_1\left(\frac{G'}{G}\right), \tag{36}$$

$$v(\xi) = \lambda a_1 - \omega + 2a_1\left(\frac{G'}{G}\right), \quad , \tag{37}$$

where

$$\xi = -2a_1(x + \omega t). \tag{38}$$

From Eqs.(17),(36) and (37), we deduce the following three types of traveling wave solutions:

Case1. If $M > 0$, then we have the hyperbolic solution

$$u(\xi) = \frac{1}{2}\left[a_1\sqrt{M}\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}\right) - \omega\right], \tag{39}$$

$$v(\xi) = a_1\sqrt{M}\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}\right) - \omega. \tag{40}$$

Case2. If $M < 0$, then we have the trigonometric solution

$$u(\xi) = \frac{1}{2}\left[a_1\sqrt{-M}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{-M}\,\xi\right) + B\cos\left(\frac{1}{2}\sqrt{-M}\,\xi\right)}{A\cos\left(\frac{1}{2}\sqrt{-M}\,\xi\right) + B\sin\left(\frac{1}{2}\sqrt{-M}\,\xi\right)}\right) - \omega\right], \tag{41}$$

$$v(\xi) = a_1\sqrt{-M}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{-M}\,\xi\right) + B\cos\left(\frac{1}{2}\sqrt{-M}\,\xi\right)}{A\cos\left(\frac{1}{2}\sqrt{-M}\,\xi\right) + B\sin\left(\frac{1}{2}\sqrt{-M}\,\xi\right)}\right) - \omega. \tag{42}$$

Case3. If $M = 0$, then we have the rational solution

$$u(\xi) = a_1\left(\frac{B}{B\xi + A}\right) - \frac{\omega}{2}, \tag{43}$$

$$v(\xi) = 2a_1\left(\frac{B}{B\xi + A}\right) - \omega. \tag{44}$$

In particular if $B = 0,\; A \neq 0,\; \lambda > 0$ and $\mu = 0$, then we deduce from Eqs.(39) and (40) that:

$$u(\xi) = \frac{1}{2}\left[a_1\lambda\coth\left(\frac{\lambda}{2}\xi\right) - \omega\right], \tag{45}$$

$$v(\xi) = a_1\lambda\coth\left(\frac{\lambda}{2}\xi\right) - \omega, \tag{46}$$

While, if $B \neq 0,\; A^2 > B^2,\; \lambda > 0 \; and \; \mu = 0$, then we deduce that:

$$u(\xi) = \frac{1}{2}\left[a_1\lambda\tanh\left(\frac{\lambda}{2}\xi + \xi_0\right) - \omega\right], \tag{47}$$

$$v(\xi) = a_1\lambda \tanh\left(\frac{\lambda}{2}\xi + \xi_0\right) - \omega, \tag{48}$$

Where $\xi_0 = \tanh^{-1}\left(\frac{A}{B}\right)$. The solutions (45)- (48) represent the solitary wave solutions of Eqs.(23) and (24).

3.3. Example 3. The symmetrically coupled KdV equations

In this subsection, we consider the symmetrically coupled KdV equations [31] in the forms:

$$u_t = u_{xxx} + v_{xxx} + 6uu_x + 4uv_x + 2u_x v = 0, \tag{49}$$

$$v_t = u_{xxx} + v_{xxx} + 6vv_x + 4vu_x + 2v_x u = 0. \tag{50}$$

The traveling wave variable (25) permits us converting Eqs.(49) and (50) into the following ODEs:

$$-\omega u' + k^2(u''' + v''') + 6uu' + 4uv' + 2u'v = 0, \tag{51}$$

$$-\omega v' + k^2(u''' + v''') + 6vv' + 4vu' + 2v'u = 0. \tag{52}$$

Considering the homogeneous balance between highest order derivatives and nonlinear terms in Eqs.(51) and (52), we have

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \qquad a_2 \neq 0, \tag{53}$$

$$v(\xi) = b_0 + b_1\left(\frac{G'}{G}\right) + b_2\left(\frac{G'}{G}\right)^2, \qquad b_2 \neq 0, \tag{54}$$

Where a_0, a_1, a_2, b_0, b_1 and b_2 are arbitrary constants to be determined later. Substituting Eqs.(53) and (54) with Eq.(5) into Eqs.(51) and (52), collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ and setting them to zero, we have the following system of algebraic equations:

$\left(\frac{G'}{G}\right)^0$: $-k^2\lambda^2\mu a_1 - 2k^2\mu^2 a_1 + \mu\omega a_1 - 6\mu a_0 a_1 - 6k^2\lambda\mu^2 a_2 - k^2\lambda^2\mu b_1 - 2k^2\mu^2 b_1 - 4\mu a_0 b_1 - 6k^2\lambda\mu^2 b_2 = 0,$

$\left(\frac{G'}{G}\right)^1$: $-k^2\lambda^3 a_1 - 8k^2\lambda\mu a_1 + \lambda\omega a_1 - 6\lambda a_0 a_1 - 6\mu a_1^2 - 14k^2\lambda^2\mu a_2 - 16k^2\mu^2 a_2 + 2\mu\omega a_2 - 12\mu a_0 a_2 - k^2\lambda^3 b_1 - 8k^2\lambda\mu b_1 - 4\lambda a_0 b_1 - 6\mu a_1 b_1 - 14k^2\lambda^2\mu b_2 - 16k^2\mu^2 b_2 - 8\mu a_0 b_2 = 0,$

$\left(\frac{G'}{G}\right)^2$: $-7k^2\lambda^2 a_1 - 8k^2\mu a_1 + \omega a_1 - 6a_0 a_1 - 6\lambda a_1^2 - 8k^2\lambda^3 a_2 - 52k^2\lambda\mu a_2 + 2\lambda\omega a_2 - 12\lambda a_0 a_2 - 18\mu a_1 a_2 - 7k^2\lambda^2 b_1 - 8k^2\mu b_1 - 4a_0 b_1 - 6\lambda a_1 b_1 - 8\mu a_2 b_1 - 8k^2\lambda^3 b_2 - 52k^2\lambda\mu b_2 - 8\lambda a_0 b_2 - 10\mu a_1 b_2 = 0,$

$\left(\frac{G'}{G}\right)^3$: $-12k^2\lambda a_1 - 6a_1^2 - 38k^2\lambda^2 a_2 - 40k^2\mu a_2 + 2\omega a_2 - 12a_0 a_2 - 18\lambda a_1 a_2 - 12\mu a_2^2 - 12k^2\lambda b_1 - 6a_1 b_1 - 8\lambda a_2 b_1 - 38k^2\lambda^2 b_2 - 40k^2\mu b_2 - 8a_0 b_2 - 10\lambda a_1 b_2 - 12\mu a_2 b_2 = 0,$

$\left(\frac{G'}{G}\right)^4$: $-6k^2 a_1 - 54k^2\lambda a_2 - 18a_1 a_2 - 12\lambda a_2^2 - 6k^2 b_1 - 8a_2 b_1 - 54k^2\lambda b_2 - 10a_1 b_2 - 12\lambda a_2 b_2 = 0,$

$\left(\frac{G'}{G}\right)^5$: $-24k^2 a_2 - 12a_2^2 - 24k^2 b_2 - 12a_2 b_2 = 0,$

$\left(\frac{G'}{G}\right)^0$: $-k^2\lambda^2\mu a_1 - 2k^2\mu^2 a_1 - 6k^2\lambda\mu^2 a_2 - k^2\lambda^2\mu b_1 - 2k^2\mu^2 b_1 + \mu\omega b_1 - 2\mu a_0 b_1 - 6k^2\lambda\mu^2 b_2 = 0,$

$\left(\frac{G'}{G}\right)^1$: $-k^2\lambda^3 a_1 - 8k^2\lambda\mu a_1 - 14k^2\lambda^2\mu a_2 - 16k^2\mu^2 a_2 - k^2\lambda^3 b_1 - 8k^2\lambda\mu b_1 + \lambda\omega b_1 - 2\lambda a_0 b_1 - 6\mu a_1 b_1 - 6\mu b_1^2 - 14k^2\lambda^2\mu b_2 - 16k^2\mu^2 b_2 + 2\mu\omega b_2 - 4\mu a_0 b_2 = 0,$

$\left(\frac{G'}{G}\right)^2$: $-7k^2\lambda^2 a_1 - 8k^2\mu a_1 - 8k^2\lambda^3 a_2 - 52k^2\lambda\mu a_2 - 7k^2\lambda^2 b_1 - 8k^2\mu b_1 + \omega b_1 - 2a_0 b_1 - 6\lambda a_1 b_1 - 10\mu a_2 b_1 - 6\lambda b_1^2 - 8k^2\lambda^3 b_2 - 52k^2\lambda\mu b_2 + 2\lambda\omega b_2 - 4\lambda a_0 b_2 - 8\mu a_1 b_2 - 18\mu b_1 b_2 = 0,$

$\left(\frac{G'}{G}\right)^3$: $-12k^2\lambda a_1 - 38k^2\lambda^2 a_2 - 40k^2\mu a_2 - 12k^2\lambda b_1 - 6a_1 b_1 - 10\lambda a_2 b_1 - 6b_1^2 - 38k^2\lambda^2 b_2 - 40k^2\mu b_2 + 2\omega b_2 - 4a_0 b_2 - 8\lambda a_1 b_2 - 12\mu a_2 b_2 - 18\lambda b_1 b_2 - 12\mu b_2^2 = 0,$

$\left(\frac{G'}{G}\right)^4$: $-6k^2 a_1 - 54k^2\lambda a_2 - 6k^2 b_1 - 10a_2 b_1 - 54k^2\lambda b_2 - 8a_1 b_2 - 12\lambda a_2 b_2 - 18b_1 b_2 - 12\lambda b_2^2 = 0,$

$\left(\frac{G'}{G}\right)^5$: $-24k^2 a_2 - 24k^2 b_2 - 12a_2 b_2 - 12b_2^2 = 0.$ (55)

Solving the above algebraic Eqs.(55) by using the Maple or Mathematica, we have

$$a_0 = b_0 = 0, \quad a_1 = -\frac{\omega\lambda}{\lambda^2 + 8\mu}, \quad a_2 = -\frac{\omega}{\lambda^2 + 8\mu}, \quad b_1 = -\frac{\omega\lambda}{\lambda^2 + 8\mu},$$
$$b_2 = -\frac{\omega}{\lambda^2 + 8\mu}, \quad k = \pm\sqrt{\frac{\omega}{2\lambda^2 + 16\mu}}. \tag{56}$$

Substituting Eq.(56) into Eqs.(53) and (54) yields

$$u(\xi) = -\frac{\omega\lambda}{\lambda^2 + 8\mu}\left(\frac{G'}{G}\right) - \frac{\omega}{\lambda^2 + 8\mu}\left(\frac{G'}{G}\right)^2, \tag{57}$$

$$v(\xi) = -\frac{\omega\lambda}{\lambda^2 + 8\mu_1}\left(\frac{G'}{G}\right) - \frac{\omega}{\lambda^2 + 8\mu}\left(\frac{G'}{G}\right)^2, \tag{58}$$

Where

$$\xi = \pm\sqrt{\frac{\omega}{2\lambda^2 + 16\mu}}(x + \omega t). \tag{59}$$

From Eqs. (17), (57) and (58), we deduce the following three types of traveling wave solutions:

Case 1. If $M > 0$, Then we have the hyperbolic solution

$$u(\xi) = \frac{\omega}{4(\lambda^2 + 8\mu)}\left[\lambda^2 - M\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}\right)^2\right], \tag{60}$$

$$v(\xi) = \frac{\omega}{4(\lambda^2 + 8\mu)}\left[\lambda^2 - M\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}\right)^2\right]. \tag{61}$$

Case2. If $M < 0$, Then we have the trigonometric solution

$$u(\xi) = \frac{\omega}{4(\lambda^2 + 8\mu)}\left[\lambda^2 + M\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}\right)^2\right], \tag{62}$$

$$v(\xi) = \frac{\omega}{4(\lambda^2 + 8\mu)}\left[\lambda^2 + M\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{M}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{M}\,\xi\right)}\right)^2\right]. \tag{63}$$

Case3. If $M = 0$, Then we have the rational solution

$$u(\xi) = \frac{\omega}{4(\lambda^2 + 8\mu)}\left[\lambda^2 - 4\left(\frac{B}{B\xi + \mathrm{A}}\right)^2\right], \tag{64}$$

$$v(\xi) = \frac{\omega}{4(\lambda^2 + 8\mu)}\left[\lambda^2 - 4\left(\frac{B}{B\xi + \mathrm{A}}\right)^2\right]. \tag{65}$$

In particular if $B = 0$, $A \neq 0$, $\lambda > 0$ and $\mu = 0$, then we deduce from Eq.(60) and Eq.(61) that:

$$u(\xi) = \frac{-\omega}{4}\operatorname{csch}^2\left(\frac{\lambda}{2}\xi\right), \tag{66}$$

$$v(\xi) = \frac{-\omega}{4}\operatorname{csch}^2\left(\frac{\lambda}{2}\xi\right), \tag{67}$$

while, if $B \neq 0$, $A^2 > B^2$, $\lambda > 0$ *and* $\mu = 0$, then we deduce that:

$$u(\xi) = \frac{\omega}{4}\operatorname{sech}^2\left(\frac{\lambda}{2}\xi + +\xi_0\right), \tag{68}$$

$$v(\xi) = \frac{\omega}{4}\operatorname{sech}^2\left(\frac{\lambda}{2}\xi + +\xi_0\right), \tag{69}$$

Where $\xi_0 = \tanh^{-1}\left(\frac{A}{B}\right)$. The solutions (66) - (69) represent the solitary wave solutions of Eqs. (49) and (50).

IV. Conclusion

In this work, we have seen that three types of traveling wave solutions in terms of hyperbolic, trigonometric and rational functions for the vcKdV equation, the MDWW equations and the symmetrically coupled KdV equations are successfully found out by using the $\left(\frac{G'}{G}\right)$-expansion method. From our results obtained in this paper, we conclude that the $\left(\frac{G'}{G}\right)$-expansion method is powerful, effective and convenient. The performance of this method is reliable, simple and gives many new solutions. The $\left(\frac{G'}{G}\right)$-expansion method has more advantages: It is direct and concise. Also, the solutions of the proposed nonlinear evolution equations in this paper have many potential applications in physics and engineering. Finally, this method provides a powerful mathematical tool to obtain more general exact solutions of a great many nonlinear PDEs in mathematical physics.

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