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A Study of Periodic Points and Their Stability on a One-Dimensional Chaotic System

Dr. Nabajyoti Das

Assistant Professor, Department of Mathematics Jawaharlal Nehru College, Boko District: Kampur, State: Assam, Country: India

Abstract: This paper highlights three objectives of the quadratic iterator

$$x_{n+1} = F(x_n) = ax_n^2 - bx_n, \quad n = 0,1,2,\ldots$$

Where $x_n \in [0, 4]$, a and b and are tunable parameters. Firstly, by adopting suitable numerical methods and computer programs we evaluate the period-doubling: $1 \to 2 \to 4 \to 8 \to \ldots$ bifurcations, a route from order into chaos. Secondly, we analyze the stability of periodic points. Thirdly, we draw the bifurcation diagram in order to support our period doubling orbits and chaotic region, and some illuminating results are obtained as the measure of chaos.

Key Words: Period-doubling bifurcation/ Periodic orbits / Chaos / Bifurcation diagram/ Stability of periodic points.
2010 Classification: 37 G 15, 37 G 35, 37 C 45

I. Introduction

Period-doubling: $period\ 1 \to period\ 2 \to period\ 4 \to \ldots \to\ period\ 2^k \to \ldots \to chaos$

Bifurcations, as a universal route to chaos, is one of the most exciting discoveries of the last few years in the field of nonlinear dynamical systems. This remarkable discovery was made by the theoretical physicist Mitchell J. Feigenbaum in the mid-1970s. The initial universality discovered by Feigenbaum in one-dimensional iterations with the logistic map, $x_{n+1} = \lambda x_n(1 - x_n)$ and the trigonometric sine function, $x_{n+1} = A\sin(\pi x_n)$ has successfully led to discover that large classes of nonlinear systems exhibit transitions to chaos which are universal and quantitatively measurable.

If S be a suitable function space and H, the hyper surface of co-dimension 1 that consists of the maps in S having derivative –1 at the fixed point, then the Feigenbaum universality is closely related to the doubling operator, T acting in S defined by

$$(Tf)(x) = -\alpha f(f(\alpha^{-1}x)), \quad f \in S$$

Where α=2.5029078750957…, a universal scaling factor. The principal properties of T that lead to universality are

(i) T has a fixed point 'g' ;

(ii) The linearized transformation DT (g) has only one eigenvalue δ greater than 1 in modulus; here δ= 4.6692016091029…

(iii) The unstable manifold corresponding to δ intersects the surface H transversally. In

One dimensional case, these properties have been proved by Lanford [8]. Furthermore, one of his fascinating discoveries is that if a family f presents period doubling bifurcations then there is an infinite sequence $\{b_n\}$ of bifurcation values such that

$$\lim_{n\to\infty} \frac{b_n - b_{n-1}}{b_{n+1} - b_n} = \delta,$$

Where δa universal number is already mentioned above. Moreover, his observation suggests that there might be a universal size-scaling in the period doubling sequence designated as the Feigenbaum α value

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = 2.5029\ldots$$

Where d_n is the size of the bifurcation pattern of period 2^n just before it gives birth to period 2^{n+1}? The birth and flowering of the Feigenbaum universality with numerous nonlinear models has motivated our research enterprise [1, 6, 7, 9, and 10].

We now highlight some useful concepts which are absolutely useful for our purpose.

1.1 Discrete dynamical systems

Any $C^k (k \geq 1)$ map $E : U \to \Re^n$ on the open set $U \subset R^n$ defines an n-dimensional **discrete-time** (autonomous) smooth dynamical system by the state equation

$$\bar{x}_{t+1} = E(\bar{x}_t), t = 1,2,3,\ldots.. \qquad (1.1)$$

Where $\overline{x}_t \in \Re^n$ is the state of the system at time t and E maps $\overline{x}_t$ to the next state $\overline{x}_{t+1}$. Starting with an initial data $\overline{x}_0$, repeated applications (iterates) of E generate a discrete set of points (the orbits) $\{E^t(\overline{x}_0) : t = 0,1,2,3,.....\}$, where $E^t(\overline{x}) = \underbrace{E \circ E \circ ... \circ E(\overline{x})}_{t\text{ times}}$ [9].

1.2 Definition: A point $\overline{x}^* \in \Re^n$ is called a **fixed point** of E if $E^m(\overline{x}^*) = \overline{x}^*$, for all $m \in$ ☪.

1.3 Definition: A point $\overline{x}^* \in \Re^n$ is called a **periodic point** of E if $E^q(\overline{x}^*) = \overline{x}^*$, for some integer $q \geq 1$.

1.4 Definition: The closed set $\boldsymbol{A} \in \Re^n$ is called the attractor of the system $\overline{x}_{t+1} = E(\overline{x}_t)$, if (i) there exists an open set $\boldsymbol{A}_0 \supset \boldsymbol{A}$ such that all trajectories $\overline{x}_t$ of system beginning in $\boldsymbol{A}_0$ are definite for all $t \geq 0$ and tend to $\boldsymbol{A}$ for $t \to \infty$, that is, $\operatorname{dist}(\overline{x}_t, \boldsymbol{A}) \to 0$ for $t \to \infty$, if $\overline{x}_0 \in \boldsymbol{A}_0$, where $\operatorname{dist}(\overline{x}, \boldsymbol{A}) = \inf_{\overline{y} \in \boldsymbol{A}} \|\overline{x} - \overline{y}\|$ is the distance from the point $\overline{x}$ to the set $\boldsymbol{A}$, and (ii) no eigensubset of $\boldsymbol{A}$ has this property.

1.5 Definition: A system is called chaotic if it has at least one chaotic attractor.

1.6 Diffeomorphism: Let A and B are open subsets of $\Re^n$. A map $E : A \to B$ is said to be a **diffeomorphism** if it is a bijection and both E and E^{-1} are differentiable mapping. E is called a $C^k -$ differentiable if both E and E^{-1} is $C^k -$ maps.

1.7: Stability Theorem: A sufficient condition for a periodic point $\overline{x}$ of period q for a diffeomorphism $E : \Re^n \to \Re^n$ to be stable is that the eigenvalues of the derivative $DE^q(\overline{x})$ are less than one in absolute value.

Armed with all these ideas and concepts, we now proceed to concentrate to our main aim and objectives

II. Feigenbaum Universality

We consider a one-dimensional map of the interval $x_{n+1} = F(x_n) \equiv F(x_n, b)$, where b is a control parameter. We are interested in the maps with quadratic maxima where, as the key system parameter b increases, a stable fixed point gives birth to a stable 2-cycle, which then gives birth to a stable 4-cycle, and so on until at $b = b_\infty$ all cycles of order 2^n are unstable and the invariant set of the map consists of 2^∞ points. Our model belongs to this category.

The condition for a fixed point is that $x^* = F(x^*, b)$ corresponding to a one-cycle. In order to decide the stability of the fixed point, we set $\delta x = x - x^*$ and study the approximate linear map $\delta x_{n+1} = F'(x^*, b)\delta x_n$

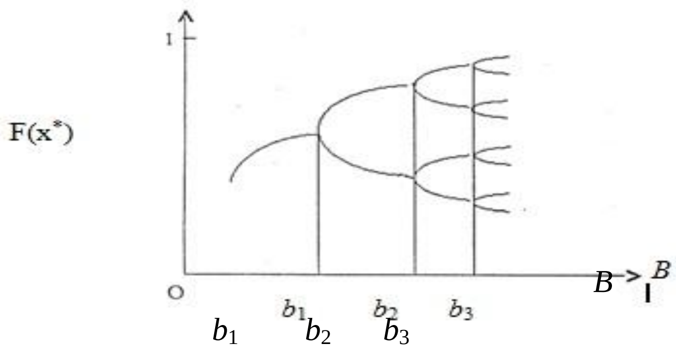

Fig2.1 Feigenbaum tree from order to chaos

Whose solution is? $$\delta x_n = F'(x^*, b)^{n-1}\delta x_0$$

A fixed point is stable if $\left|F'(x^*, b)\right| < 1$ an unstable if $\left|F'(x^*, b)\right| > 1$ and the value b_1 where $F'(x^*, b_1) = -1$ signals a bifurcation. After the value $b = b_1$ is passed, the point $x^*(b)$ becomes unstable and there appear around it two points $x_{11}^*(b)$ and $x_{12}^*(b)$ forming a stable periodic trajectory of period-2. The differences

$$\left|x_{11}^*(b) - x^*(b_1)\right| \text{ And } \left|x_{12}^*(b) - x^*(b_1)\right|$$

Are of order $(b-b_1)^{1/2}$ as $(b-b_1)\to 0$, while $\left|x^*(b)-x^*(b_1)\right| = o(b-b_1)$. thus, under period-doubling bifurcations the previously stable fixed point becomes unstable, and a stable periodic trajectory of period-2 appears near it.

As the parameter is further increased, the original fixed point continues to exist as an unstable fixed point, and all the remaining points are attracted towards the stable periodic trajectory of period-2. This happens upto some value $b = b_2$ at which the periodic trajectory of period-2 loses stability in such a way that

$$\left.\frac{dF^2(x,b_1)}{dx}\right|_{x=x_{11}^*} = \left.\frac{dF^2(x,b_2)}{dx}\right|_{x=x_{12}^*} = -1$$

We can then repeat the same arguments and find that the periodic trajectory of period-2 becomes unstable and a periodic trajectory of period-4 appears near it. As b is continuously increased, an infinite sequence $\{b_n\}$ of parameter values emerges such that at $b = b_n$ there is a loss of stability of the periodic trajectory of period 2^{n-1} and a periodic Trajectory of period 2^n arises. By now, we can imagine what happens when

$$b = b_\infty = \lim_{n\to\infty} b_n;$$

The map $F(x,b_\infty)$ has an invariant set, say ***A*** Cantor's type surrounded by infinitely many unstable periodic trajectories of periods 2^n. Moreover, all the points except those belonging to these unstable trajectories and their inverse images are attracted to ***A*** under the action of $F(x,b_\infty)$. Feigenbaum universality in its simplest form means that the sequence $\{b_n\}$ behaves in a universal manner, that is, $b_\infty - b_n \approx C\delta^{-n}$, where the constant C depends on the family F, while δ is the Feigenbaum universal constant. Moreover, the structure of the attractor ***A***, in particular, its Housedroff dimension, and the behaviour of the iterates F^n in a neighborhood of b = b_{∞} do not depend on ***A*** [2--5].

III. Feigenbaum δ (delta) to make predictions

One possible and very useful interpretation of the universality of the Feigenbaum constant δ would be by using it for quantitative prediction. At a more practical level, the existence of δ allows us to make quantitative predictions about the behavior of a nonlinear system, even if we cannot solve the equations describing the system. More importantly, this is true even if we do not know what the fundamental equations for the system are, as is often the case. For example, if we observe that a particular system undergoes a period-doubling bifurcation from period-1 to period-2 at a parameter value b_1, and from period-2 to period-4 at a value b_2, then we can use δ to predict that the system will make a transition from period-4 to period-8 at b_3 given by

$$b_3 \approx \frac{b_2 - b_1}{\delta} + b_2 \tag{1.1}$$

Observing the first two period-doublings produces no guarantee that a third will occur, but if it does occur, equation (1.1) gives us a reasonable prediction of the parameter value near

Which we should look to see the transitions. Similarly $b_4 \approx \dfrac{b_3 - b_2}{\delta} + b_3$, and so on [10].

IV. Numerical method for obtaining periodic points

To find a periodic point of our model we apply the Newton-Recurrence formula:

$$x_{n+1} = x_n - \frac{1}{\left[\frac{d}{dx}g(x_n)\right]}g(x_n), \qquad \text{where } n = 1,2,3,\ldots$$

[We later see that this map g equals Fn-I, where I is the identity function]

The Newton formula actually gives the zero(s) of a map, and to apply this numerical tool, one needs a number of recurrence formulae which are given below:

Let the initial value of x be x_0. Then

$$F(x_0) = ax_0^2 - bx_0) = x_1(say),$$

$$F^2(x_0) = F(F(x_0)) = F(x_1) = ax_1^2 - bx_1) = x_2(say),$$

Proceeding in this manner, the following recurrence formula can be established:

$$x_n = ax_{n-1}^2 - bx_{n-1}, \qquad n = 1, 2, 3, ...$$

V. Numerical Method for Finding Bifurcation Values

The derivative of F^n can be obtained as follows:

$$\left.\frac{dF}{dx}\right|_{x=x_0} = 2ax_0 - b$$

Again, by the chain rule of differentiation we get

$$\left.\frac{d(F^2)}{dx}\right|_{x=x_0} = \left.\frac{dF}{dx}\right|_{F(x_0)} \left.\frac{dF}{dx}\right|_{x=x_0} = (2ax_1 - b)(2ax_0 - b), \quad \text{where } x_1 = F(x_0)$$

Proceeding in this way we can obtain

$$\left.\frac{d(F^n)}{dx}\right|_{x=x_0} = (2ax_{n-1} - b)........(2ax_0 - b)$$

We recall that the value of b will be the bifurcation value for the map F^n when its derivative $\frac{d(F^n)}{dx}$ at a periodic point equals *-1*. Also the Feigenbaum theory says that

$$b_{n+2} \approx b_{n+1} + \frac{b_{n+1} - b_n}{\delta}, \quad \text{where } n = 1, 2, 3, ... \tag{1.2}$$

and δ is the Feigenbaum universal constant. We notice that if we put

$$I = \frac{dF^n}{dx} + 1,$$

Then I turn out to be a function of the parameter b. The bifurcation value of the parameter b of the period n occurs when $I(b)$ equals zero. This means, in order to find a bifurcation value of period n, one needs the zero of the function $I(b)$, which is given by the Secant method, applied to the function $I(b)$ which is given by

$$b_{n+1} = b_n - \frac{I(b_n)(b_n - b_{n-1})}{I(b_n) - I(b_{n-1})} \tag{1.3}$$

This method depends very sensitively on the initial condition.

VI. Our model and associated universal results

Our concerned model is

$$F(x) = ax^2 - bx \tag{1.4}$$

Where $x \in [0,4]$, a and b are tunable parameters. To find points of period-one, it is necessary to solve the equation given by $F(x) = x$ which gives the points that satisfy the condition $x_{n+1} = x_n$ for all n. The solutions are $x_1^* = 0$ and $x_2^* = (1+b)/a$. If we fix $a = -1$, the function F is maximum at $x = -b/2$ and its maximum value equals $b^2/4$. Taking this as 4, we have $b = \pm 4$. We again fix $b \in [-4, -1[$ for our purpose. In this case, the fixed points of F are the intersection of the graphs of $y = F(x)$ and $y = x$.

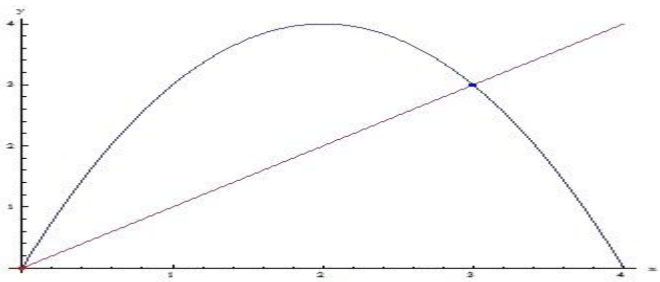

Figure 1.1 Graphs of y = F(x) and y = x for b = -4

The periodic points x_1^* (Red) and x_2^* (Blue) are shown in the figure (1.1). The stability of the critical points may be determined using the following theorem:

Using stability theorem, we have $\left|\frac{d}{dx}F(x_1^*)\right| = |-b| > 1$. Thus the fixed point $x_1^* = 0$ is always unstable for all $b \in [-4,-1[$ and $\left|\frac{d}{dx}F(x_2^*)\right| = |b+2|$, the point $x_2^* = -1-b$ stable for $-3 < b < -1$. For example, if we take the parameter value *b* = -2.9, then the orbit generated by the initial point $x_0 = 1.5$ attracted to the fixed point $x_2^* = 1.9$ in the figure 1.2.

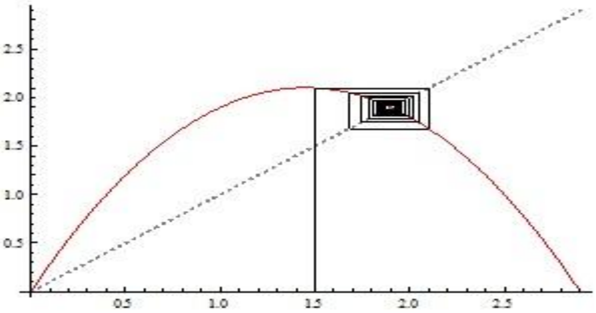

Figure 1.2 Staircase for the initial point $x_0 = 1.5$ and parameter b = -2.9

Having studied the dynamics of the quadratic iterator F in detail for parameter values between -1 and -3, we continue to decrease *b* beyond -3. For such small parameter values the fixed point x_2^* is not stable anymore, it is unstable. Hence, the **first bifurcation value** is $b_1 = -3$. To find points of period 2, we consider the iterated map $F^2(x)$. Here,

$$F^2(x) = -b(-bx - x^2) - (-bx + x^2)^2$$

The periodic points of $F^2(x)$ are given by the equation

$$F^2(x) = x \tag{1.5}$$

Which gives $x = x_1^*, x_2^*, x_{11}^*, x_{12}^*$, where

$$x_1^* = 0, x_2^* = -1-b, x_{11}^* = 1/2(1-b-\sqrt{-3+2b+b^2}), x_{12}^* = 1/2(1-b+\sqrt{-3+2b+b^2})$$

These four points are the intersection of the graphs of $y = F^2(x)$ and *y* = *x*, in the figure 1.3. The periodic points x_1^*, x_2^*, x_{11}^* and x_{12}^* are marked as *A1, A2, B1* and *B2.*

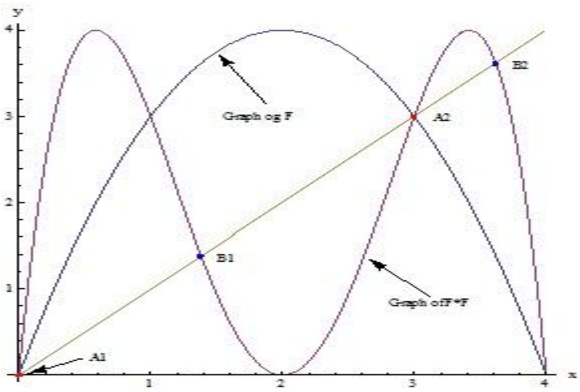

Figure 1.3 Graphs of $y = F^2(x)$ and y = x for b= -4

Stability of the first two fixed points is already discussed. Let us discuss the stability of the new points: x_{11}^* and x_{12}^*. We note that these new solutions are defined only for $b \le -3$. Moreover, at $b = -3$, $x_{11}^* = x_{12}^* = \frac{1}{2}(1-b)$, i.e., these two solutions bifurcate from the fixed point x_2^*. The points x_{11}^* and x_{12}^* form a two-cycle, one being the image of the other. Thus, at parameter $b = -3$, our map orbits undergo period-doubling bifurcations. Just above $b = -3$ the orbits converge to a single value of x. Just below $b = -3$, the orbits tend to this alteration between two values of x.

Let us see how the derivatives of the map function $F(x)$ and of the second iterate function $F^2(x)$ change at the bifurcation value. The equation:

$$\left.\frac{dF(x)}{dx}\right|_{x=x_2^*} = b+2 \qquad (1.6)$$

tells us that function $\frac{dF(x)}{dx}$ passes through the value -1 as *b* decreases through -3. Next we can evaluate the derivative of the second iterate function by using the chain-rule of differentiation:

$$\frac{dF^2(x)}{dx} = \frac{d}{dx}[F(F(x))] = \left.\frac{dF}{dx}\right|_{F(x)} \left.\frac{dF}{dx}\right|_{x}$$

If we now evaluate the derivative at one of the above two new fixed points, say x_{11}^*, then we find

$$\left.\frac{dF^2(x)}{dx}\right|_{x_{11}^*} = \left.\frac{dF}{dx}\right|_{x_{12}^*} \left.\frac{dF}{dx}\right|_{x_{11}^*} = \left.\frac{dF^2(x)}{dx}\right|_{x_{12}^*} \qquad (1.7)$$

In arriving at the last result, we made use of $x_{12}^* = F(x_{11}^*)$ for the two fixed points. Equation (1.7) states a rather surprising and important result—

The derivative of $F^2(x)$ are the same at both the fixed points that are actually part of the two-cycle. This result implies that both of these fixed points are either attracting or both are repelling, and that they have the same 'degree' of stability or instability. Again, since the derivative of $F(x)$ equals -1 for the parameter $b=-3$, equation (1.7) tells us that the derivative of $F^2(x)$ equals +1 for $b=-3$. As *b* decreases further, the derivative of $F^2(x)$ increases and the fixed points become stable. Besides, the unstable fixed point of $F(x)$ located at x_2^* is also an unstable fixed point of $F^2(x)$.

The 2-cycle fixed points of $F^2(x)$ continue to be stable fixed points until parameter value $b_2 = -3.449489742783...$ we have values of x_{11}^* and x_{12}^* as *1.5176380902051063*…. and *2.931851652577893*…. respectively at $b_2 = -3.449489742783...$ Also for this value of b

$$\left.\frac{dF^2(x)}{dx}\right|_{x\,=\,1.5176380902051063..} = -1, \qquad \left.\frac{dF^2(x)}{dx}\right|_{x\,=\,2.931851652577893....} = -1$$

The above results guarantee that if a system is stable or unstable at a periodic point, then the system is so at any other periodic point. So our study will be completed if we study the dynamics at any of the periodic points. We can find that for values of b smaller than b_2, the derivative is more negative than -1. Hence for b values smaller than b_2, the 2-cycle points are repelling fixed points. We find that for values just smaller than b_2, the orbits settle into a 4-cycle, that is, the orbit cycles among 4 values which we can label as

$$x_{21}^*, x_{22}^*, \ x_{23}^*, x_{24}^*.$$

These points are nothing but the intersection of the graphs of $y=F^4(x)$ and $y=x$. To determine these periodic points analytically, we need to solve an eight degree equation, namely $F^4(x)=x$ which is manually cumbersome and time consuming. Therefore, for finding periodic points, bifurcation values of F^4 as well as for higher iterated map functions, we have to write a computer program. We write here a C-program for our purpose.

Using the relation (1.2), an approximate value b_3' of *b* is obtained. Since the Secand method needs two initial values, we use b_3 and a slightly larse value, say b_3+10^{-4} as the two initial values to apply this method and ultimately obtain b_3. In the like manner, the same procedure is employed to obtain the successive bifurcation values b_4, b_5, etc. to oue requirement. Through our numerical mechanism, we obtain some periodic points and bifurcation values. In the Table 1.1 Period doubling cascades are shown:

VII. Period doubling cascade

Table 1.1

Period	**One of the Periodic points**	**Bifurcation Points.**
1	$x_1 = 2.000000000000\ldots$	$b_1 = -3.000000000000\ldots$
2	$x_2 = 1.517638090205\ldots$	$b_2 = -3.449489742783\ldots$
4	$x_3 = 2.905392825125\ldots$	$b_3 = -3.544090359552\ldots$
8	$x_4 = 3.138826940664\ldots$	$b_4 = -3.564407266095\ldots$
16	$x_5 = 1.241736888630\ldots$	$b_5 = -3.568759419544\ldots$
32	$x_6 = 3.178136193507\ldots$	$b_6 = -3.569691609801\ldots$
64	$x_7 = 3.178152098553\ldots$	$b_7 = -3.569891259378\ldots$
128	$x_8 = 3.178158223315\ldots$	$b_8 = -3.569934018374\ldots$
256	$x_9 = 3.178160120824\ldots$	$b_9 = -3.569943176048\ldots$
512	$x_{10} = 1.696110052289\ldots$	$b_{10} = -3.569945137342\ldots$
1024	$x_{11} = 1.696240778303\ldots$	$b_{11} = -3.569945557391\ldots$
...	... ...	... ...

Based on these values, the ratios of successive separations of bifurcation points are given by,

$$\frac{b_2 - b_1}{b_3 - b_2} \approx \frac{b_3 - b_2}{b_4 - b_3} \approx \frac{b_4 - b_3}{b_5 - b_4} \approx \frac{b_5 - b_4}{b_6 - b_5} \approx \frac{b_6 - b_5}{b_7 - b_6} \approx \ldots \approx \frac{b_k - b_{k-1}}{b_{k+1} - b_k}$$

And have a particular scaling associated with them. We see that

$$\delta_1 = \frac{b_2 - b_1}{b_3 - b_2} = 4.751446218163496..., \qquad \delta_2 = \frac{b_3 - b_2}{b_4 - b_3} = 4.65625101778075...,$$

$$\delta_3 = \frac{b_4 - b_3}{b_5 - b_4} = 4.66824223480187..., \qquad \delta_4 = \frac{b_5 - b_4}{b_6 - b_5} = 4.668773947278035...,$$

$$\delta_5 = \frac{b_6 - b}{b_7 - b_6} = 4.66913201146184..., \qquad \delta_6 = \frac{b_7 - b_6}{b_8 - b_7} = 4.66918299485741...,$$

$$\delta_7 = \frac{b_8 - b_7}{b_9 - b_8} = 4.66919831388462..., \qquad \delta_8 = \frac{b_9 - b_8}{b_{10} - b_9} = 4.66920002791641...,$$

$\delta_9 = \dfrac{b_{10} - b_9}{b_{11} - b_{10}} = 4.66920287863952...,$ And so on.

The ratios tend to a constant as *k* tends to infinity: more formally

$$\lim_{k \to \infty} \left[\frac{b_k - b_{k-1}}{b_{k+1} - b_k} \right] = \delta = 4.669201...$$

The nature of δ is universal i.e. it is the same for a wide range of different iterators

VIII. Bifurcation diagram

The different behaviors of a system for different values of the parameter can be qualitatively analyzed by using a bifurcation diagram, which is created by plotting the asymptotic orbits of the maps (*y* axis) generated for different values of the parameter (*x* axis). A bifurcation diagram is essentially a diagram of attractors, because almost all initial points are attracted to the points shown in the figure of our model, provided a sufficient

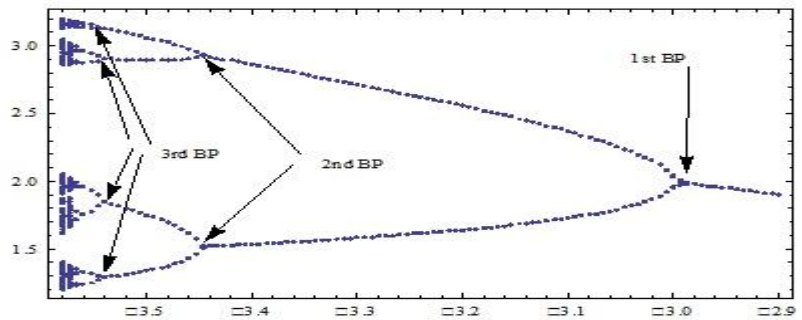

Figure 1.4 Bifurcation diagram for b in the range $-3.6 \le b \le -2.9$

Numbers of transients have been thrown away [9, 10]. Fixed points and periodic points are trivial attractor, while the darkened vertical segments are chaotic attractors. Just beyond b = *-3.96995* (approx) the system becomes chaotic. However, the system is not chaotic for all parameter values b smaller than *-3.56995* (approx). If we zoom into the details of the bifurcation diagram by changing to smaller and smaller scales both in x and in b, we see that within the chaotic region, there are many *periodic windows*, that is, lucid intervals where only periodic orbits exist instead of chaotic output.

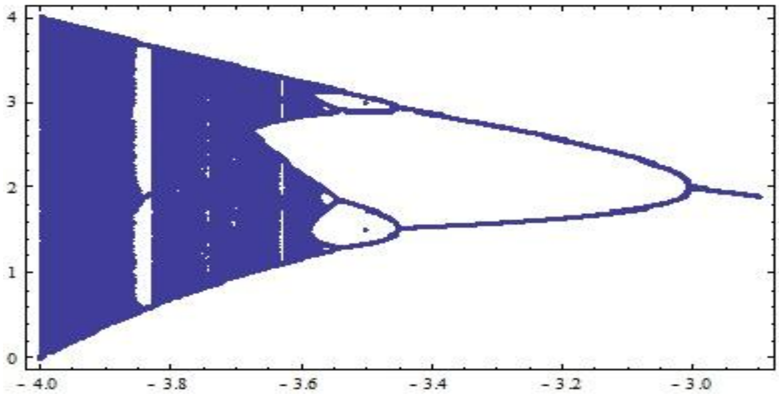

Figure 1.5 Bifurcation diagram for b in the range $-4.0 \le b \le -2.9$

IX. Conclution

The study of chaos in population models is quite interesting. Although there are so many methods for finding bifurcation values, we have developed own numerical mechanism for establishing Feigenbaum tree of bifurcation values leading to chaotic region the study of which is intrinsically marvelous. Our method seems to be applicable to all the chaotic models.

X. Acknowledgement:

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