Contribution of Axial Deformation in the Analysis of Rigidly Fixed Portal Frames

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ABSTRACT: *In this work the stiffness equations for evaluating the internal stress of rigidly fixed portal frames by the displacement method were generated. But obtaining the equations for the internal stresses required non-scalar or parametric inversion of the structure stiffness matrix. To circumvent this problem, the flexibility method was used taking advantage of the symmetrical nature of the portal frame and the method of virtual work. These were used to obtain the internal stresses on rigidly fixed portal frames for different cases of external loads when axial deformation is considered. A dimensionless constant s was used to capture the effect of axial deformation in the equations. When it is set to zero, the effect of axial deformation is ignored and the equations become the same as what can be obtained in any structural engineering textbook. These equations were used to investigate the contribution of axial deformation to the calculated internal stresses and how they vary with the ratio of the flexural rigidity of the beam and columns and the height to length ratio of the loaded portal frames.*

Keywords: *Axial deformation, flexural rigidity, flexibility method, Portal frames, stiffness matrix,*

I. INTRODUCTION

Structural frames are primarily responsible for strength and rigidity of buildings. For simpler single storey structures like warehouses, garages etc portal frames are usually adequate. It is estimated that about 50% of the hot-rolled constructional steel used in the UK is fabricated into single-storey buildings (Graham and Alan, 2007). This shows the increasing importance of this fundamental structural assemblage. The analysis of portal frames are usually done with predetermined equations obtained from structural engineering textbooks or design manuals. The equations in these texts were derived with an underlying assumption that deformation of structures due to axial forces is negligible giving rise to the need to undertake this study. The twenty first century has seen an astronomical use of computers in the analysis of structures (Samuelsson and Zienkiewi, 2006) but this has not completely eliminated the use of manual calculations form simple structures and for easy cross-checking of computer output (Hibbeler, 2006). Hence the need for the development of equations that capture the contribution of axial deformation in portal frames for different loading conditions.

II. DEVELOPMENT OF THE MODEL

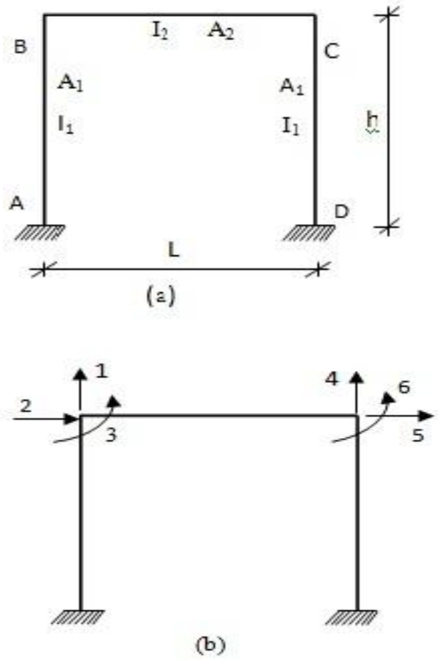


Figure 1: A simple portal frame showing its dimensions and the 6 degrees of freedom

The analysis of portal frames by the stiffness method requires the determination of the structure's degrees of freedom and the development of the structure's stiffness matrix. For the structure shown in Figure 1a, the degrees of freedom are as shown in Figure 1b. I_1 and A_1 are respectively the second moment of inertia and cross-sectional area of the columns while I_2 and A_2 are the second moment of inertia and cross-sectional area of the beam respectively. The stiffness coefficients for the various degrees of freedom considering shear deformation can be obtained from equations developed in Ghali et al (1985) and Okonkwo (2012) and are presented below

The structure's stiffness matrix can be written as:

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix}. \qquad (1)$$

Where k_{ij} is the force in coordinate (degree of freedom) i when there is a unit displacement in coordinate (degree of freedom) j. They are as follows:

$$k_{11} = \frac{EA_1}{h}$$

$$k_{12} = 0$$

$$k_{13} = \frac{6EI_2}{l^2}$$

$$k_{14} = -\frac{12EI_1}{h^3}$$

$$k_{15} = 0$$

$$k_{16} = \frac{6EI_2}{l^2}$$

$$k_{22} = \frac{12EI_1}{h^3} + \frac{EA_2}{l}$$

$$k_{23} = \frac{6EI_1}{h^2}$$

$$k_{24} = 0$$

$$k_{25} = -\frac{EA_2}{l} \quad . \quad . \quad (2)$$

$$k_{26} = 0$$

$$k_{33} = \frac{4EI_2}{l} + \frac{4EI_1}{h}$$

$$k_{34} = -\frac{6EI_2}{l^2}$$

$$k_{35} = 0$$

$$k_{36} = \frac{2EI_2}{l}$$

$$k_{44} = \frac{12EI_2}{l^3} + \frac{EA_1}{h}$$

$$k_{45} = 0$$

$$k_{46} = -\frac{6EI_2}{l^2}$$

$$k_{55} = \frac{12EI_1}{h^3} + \frac{EA_2}{l}$$

From Maxwell's Reciprocal theorem and Betti's Law $k_{ij} = k_{ji}$ (Leet and Uang, 2002).

When there are external loads on the structure on the structure there is need to calculate the forces in the restrained structure F_o as a result of the external load.

The structure's equilibrium equations are then written as

$$\{F\} = \{F_o\} + [K]\{D\} \ . \ . \ . \qquad . \qquad (3)$$

$$\{F_o\} = \begin{Bmatrix} k_{10} \\ k_{20} \\ k_{30} \\ k_{40} \\ k_{50} \\ k_{60} \end{Bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Where k_{io} is the force due to external load in coordinate i when the other degrees of freedom are restrained.

$$\{D\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Where d_i is the displacement in coordinate i.

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Where F_i is the external load with a direction coinciding with the coordinate i (McGuire et al, 2000).

By making {D} the subject of the formula in equation (3)

$$\{D\} = [K]^{-1}\{F - F_o\} \quad . \quad . \quad . \quad . \ (7)$$

Once {D} is obtained the internal stresses in the frame can be easily obtained by writing the structure's compatibility equations given as

$$M = M_r + M_1 d_1 + M_2 d_2 + M_3 d_3 \ . \qquad . \qquad (8)$$

Where M is the internal stress (bending moment) at any point on the frame, M_r is the internal stress at the point under consideration in the restrained structure while Mi is the internal stress at the point when there is a unit displacement in coordinate i.
To solve equation (8) there is need to obtain {D}. {D} can be obtained from the inversion of [K] in equation (7). Finding the inverse of K parametrically (i.e. without substituting the numerical values of E, h, l etc) is a difficult task. This problem is circumvented by using the flexibility method to solve the same problem, taking advantage of the symmetrical nature of the structure and the principle of virtual work.

III. APPLICATION OF THE FLEXIBILITY MODEL

The basic system or primary structure for the structure in Figure 1a is given in Figure 2. The removed redundant force is depicted with X_1.

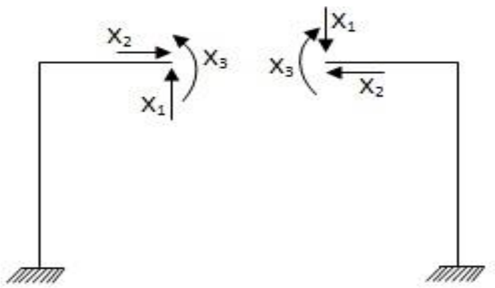


Figure 2: The Basic System showing the removed redundant forces

The flexibility matrix of the structure can be determined using the principle of virtual work.
By applying the unit load theorem the deflection in beams or frames can be determined for the combined action of the internal stresses, bending moment and axial forces with

$$D = \int \frac{\bar{M}M}{EI} ds + \int \frac{\bar{N}N}{EA} ds \qquad (9)$$

Where $\bar{M}$ and $\bar{N}$ are the virtual internal stresses while M and N are the real/actual internal stresses.

E is the modulus of elasticity of the structural material

A is the cross-sectional area of the element (McGuire et al, 2000; Nash, 1998)

If d_{ij} is the deformation in the direction of i due to a unit load at j then by evaluating equation (9) the following are obtained:

$$d_{11} = \frac{I_1 G A_1 l^3 + 6 I_2 A_1 l^2 h + 12 I_1 I_2 h}{12 E I_1 I_2 A_1} \qquad (10a)$$

$$d_{12} = 0 \qquad (10b)$$

$$d_{13} = 0 \qquad (10c)$$

$$d_{22} = \frac{2A_2 h^3 + 3I_1 l}{3EI_1 A_2} \qquad (10d)$$

$$d_{23} = \frac{-h^2}{EI_1} \qquad (10e)$$

$$d_{33} = \frac{lI_1 + 2hI_2}{EI_1 I_2} \qquad (10f)$$

From Maxwell's Reciprocal theorem and Betti's Law $d_{ij} = d_{ji}$.

The structure's compatibility equations can be written thus

$$\begin{bmatrix} d_{11} & d_{11} & d_{11} \\ d_{21} & d_{21} & d_{21} \\ d_{31} & d_{31} & d_{31} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} + \begin{Bmatrix} d_{10} \\ d_{20} \\ d_{30} \end{Bmatrix} = 0 \qquad (11a)$$

i.e. $fF + d_o = 0$. (11b)

Where F is the vector of redundant forces X_1, X_2, X_3 and d_o is the vector deformation d_{10}, d_{20}, d_{30} due to external load on the basic system (Jenkins, 1990).

$$F = f^{-1}(-d_o) \qquad (12)$$

$$f^{-1} = \frac{Adj\,[f]}{det\,[f]} \qquad (13)$$

(Stroud, 1995)

$$f^{-1} = \begin{bmatrix} \frac{1}{d_{11}} & 0 & 0 \\ 0 & \frac{d_{33}}{d_{22}d_{33} - d_{23}^2} & \frac{-d_{32}}{d_{22}d_{33} - d_{23}^2} \\ 0 & \frac{-d_{23}}{d_{22}d_{33} - d_{23}^2} & \frac{d_{22}}{d_{22}d_{33} - d_{23}^2} \end{bmatrix} \qquad (14)$$

From equations 10a – 10f

$$\frac{1}{d_{11}} = \frac{12EI_1 I_2 A_1}{I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24 I_1 I_2 h} \qquad (15a)$$

$$\frac{d_{33}}{d_{22}d_{33} - d_{23}^2} = \frac{3EI_1 A_2 (lI_1 + 2hI_2)}{2A_2 h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2} \qquad (15b)$$

$$\frac{d_{32}}{d_{22}d_{33} - d_{23}^2} = \frac{-3EI_1 I_2 G A_2 h^2}{2A_2 h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2} \qquad (15c)$$

$$\frac{d_{22}}{d_{22}d_{33} - d_{23}^2} = \frac{EI_1 I_2 (2A_2 h^3 + 3I_1 l)}{2A_2 h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2} \qquad (15d)$$

Equation (12) is evaluated to get the redundant forces and these are substituted into the structures force equilibrium (superposition) equations to obtain the internal stresses at any point.

$$M = M_o + M_1 X_1 + M_2 X_2 + M_3 X_3 \qquad (16)$$

Where M is the required stress at a point, M_o is the stress at that point for the reduced structure, M_i is stress at that point when the only the redundant force $X_i = 1$ acts on the reduced structure.
For the loaded portal frame of Figure 3, the deformations of the reduced structure due to external loads are

$$d_{10} = 0 \qquad (17a)$$

$$d_{20} = \frac{wh^2 l^2}{8EI_1} \qquad (17b)$$

$$d_{30} = \frac{-wl^2 (lI_1 + 6hI_2)}{24EI_1 I_2} \qquad (17c)$$

By substituting the values of equations (17a) – (17c) into equations (12)

$$X_1 = 0 \qquad (18a)$$

$$X_2 = \frac{-A_2 I_1 w h^2 l^3}{4(2A_2 h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \qquad (18b)$$

$$X_3 = \frac{wl^2 (2A_2 h^3 lI_1 + 3I_1^2 l^2 + 3A_2 h^4 I_2 + 18hlI_1 I_2)}{24(2A_2 h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \qquad (18c)$$

Evaluating equation (16) for different points on the structure using the force factors obtained in equations (18a) – (18c)

$$M_A = \frac{2wl^3 I_1 (h^3 A_2 - 3lI_1)}{12(2A_2 h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \qquad (19)$$

$$M_B = \frac{-2wl^3 I_1 (2h^3 A_2 + 3lI_1)}{24(2A_2 h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \qquad (20)$$

$$M_C = \frac{-2wl^3 I_1 (2h^3 A_2 + 3lI_1)}{24(2A_2 h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \qquad (21)$$

$$M_C = \frac{-wl^3 I_1 (8I_2 h^3 + s_2 l^3 I_1)}{12[4h^3 I_2 (2lI_1 + hI_2) + s_2 l^3 I_1 (lI_1 + 2hI_2)]}$$

$$M_D = \frac{2wl^3I_1(h^3A_2-3lI_1)}{12(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (22)$$

For the loaded portal frame of Figure 4, the deformations of the reduced structure due to external loads are

$$d_{10} = \frac{-wl(l^3I_1A_1+8hl^2I_2A_1+64hI_1I_2)}{128EI_1I_2A_1} \quad (23a)$$

$$d_{20} = \frac{wh^2l^2}{16EI_1} \quad (23b)$$

$$d_{30} = \frac{-wl^2(lI_1+6hI_2)}{48EI_1I_2} \quad (23c)$$

By substituting the values of equations (23a) – (23c) into equations (12)

$$X_1 = \frac{3wl(l^3I_1A_1+8hl^2I_2A_1+64hI_1I_2)}{32(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} \quad (24a)$$

$$X_2 = \frac{-2wh^2l^3A_2I_1}{16(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (24b)$$

$$X_3 = \frac{wh^3l^2A_2(3hI_2+2lI_1)+3wl^3I_1(lI_1+6hI_2)}{48(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (24c)$$

Let $\beta = \frac{s_2}{s_1}$. (25)

Evaluating equation (16) for different points on the structure using the force factors obtained in equations (24a) – (24c)

$$M_A = \frac{wl^2[h^3A_2(3hI_2+8lI_1)+3lI_1(lI_1+6hI_2)]}{48(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} - \frac{wl^4A_1(5lI_1+24hI_2)}{64(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} \quad (26)$$

$$M_B = -\frac{wl^4A_1(5lI_1+24hI_2)}{64(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} + \frac{wl^2[h^3A_2(3hI_2+2lI_1)+3lI_1(lI_1+6hI_2)]}{48(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (27)$$

$$M_C = -\frac{3wl^2(l^3I_1A_1+8hl^2I_2A_1+64hI_1I_2)}{64(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} + \frac{wl^2[h^3A_2(3hI_2+2lI_1)+3lI_1(lI_1+6hI_2)]}{48(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (28)$$

$$M_D = -\frac{3wl^2(l^3I_1A_1+8hl^2I_2A_1+64hI_1I_2)}{64(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} + \frac{wl^2[h^3A_2(3hI_2+8lI_1)+3lI_1(lI_1+6hI_2)]}{48(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (29)$$

For the loaded portal frame of Figure 5, the deformations of the reduced structure due to external loads are

$$d_{10} = -\frac{wlh^3}{12EI_1} \quad (30a)$$

$$d_{20} = \frac{wh^4}{8EI_1} \quad (30b)$$

$$d_{30} = \frac{-wh^3}{6EI_1} \quad (30c)$$

By substituting the values of equations (30a) – (30c) into equations (12)

$$X_1 = \frac{wh^3lI_2A_1}{I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2} \quad (31a)$$

$$X_2 = \frac{-wh^4A_2(3lI_1+2hI_2)}{8(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (31b)$$

$$X_3 = \frac{wh^3I_2(-h^3A_2+12lI_1)}{24(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (31c)$$

Evaluating equation (16) for different points on the structure using the force factors obtained in equations (31a) – (31c)

$$M_A = -\frac{wh^2(I_1A_1l^3+5A_1l^2hI_2+24hEI_1I_2)}{2(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} + \frac{wh^3(9h^2lI_1A_2+5A_2h^3I_2+12lI_1I_2)}{24(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (32)$$

$$M_B = \frac{wh^3l^2I_2A_1}{2(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} + \frac{wh^3I_2(-h^3A_2+12lI_1)}{24(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (33)$$

$$M_C = -\frac{wh^3l^2I_2A_1}{2(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} + \frac{wh^3I_2(-h^3A_2+12lI_1)}{24(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (34)$$

$$M_D = -\frac{wh^3l^2I_2A_1}{2(I_1A_1l^3+6I_2A_1l^2h+24hI_1I_2)} + \frac{wh^3(9h^2lI_1A_2+5A_2h^3I_2+12lI_1I_2)}{24(2A_2h^3lI_1+3I_1^2l^2+A_2h^4I_2+6hlI_1I_2)} \quad (35)$$

For the loaded portal frame of Figure 6, the deformations of the reduced structure due to external loads are

$$d_{10} = -\frac{Pa^2l}{2EI_1} \quad (36a)$$

$$d_{20} = 0 \quad (36b)$$

$$d_{30} = 0 \quad (36c)$$

By substituting the values of equations (36a) – (36c) into equations (12)

$$X_1 = \frac{6Pa^2lI_2A_1}{A_1I_1l^3+6A_1hl^2I_2+24hI_1I_2} \quad (37a)$$

$$X_2 = 0 \quad (37b)$$

$$X_3 = 0 \quad (37c)$$

Evaluating equation (16) for different points on the structure using the force factors obtained in equations (37a) – (37c)

$$M_A = -Pa\left(1 - \frac{3al^2 I_2 A_1}{A_1 I_1 l^3 + 6A_1 h l^2 I_2 + 24hI_1 I_2}\right) \quad (38)$$

$$M_B = \frac{3Pa^2 l^2 I_2 A_1}{A_1 I_1 l^3 + 6A_1 h l^2 I_2 + 24hI_1 I_2} \quad (39)$$

$$M_C = -\frac{3Pa^2 l^2 I_2 A_1}{A_1 I_1 l^3 + 6A_1 h l^2 I_2 + 24hI_1 I_2} \quad (40)$$

$$M_D = Pa\left(1 - \frac{3al^2 I_2 A_1}{A_1 I_1 l^3 + 6A_1 h l^2 I_2 + 24hI_1 I_2}\right) \quad (41)$$

For the loaded portal frame of Figure 7, the deformations of the reduced structure due to external loads are

$$d_{10} = -\frac{Pa^2 l}{4EI_1} \quad (42a)$$

$$d_{20} = \frac{Pa^2(3h-a)}{6EI_1} \quad (42b)$$

$$d_{30} = -\frac{Pa^2}{2EI_1} \quad (42c)$$

By substituting the values of equations (42a) – (42c) into equations (12)

$$X_1 = \frac{3Pa^2 l I_2 A_1}{I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2} \quad (43a)$$

$$X_2 = \frac{-Pa^2 A_2[lI_1(3h-a) + hI_2(3h-2a)]}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (43b)$$

$$X_3 = \frac{-Pa^2 I_2(-h^3 A_2 + ah^2 A_2 + 3lI_1)}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (43c)$$

Evaluating equation (16) for different points on the structure using the force factors obtained in equations (43a) – (43c)

$$M_A = -Pa + \frac{3Pa^2 l^2 I_2 A_1}{2(I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2)} + \frac{Pa^2[A_2 I_1 l(3h^2 - ah) + I_2(2A_2h^3 - aA_2h^2 + 3I_1 l)]}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (44)$$

$$M_B = \frac{3Pa^2 l^2 I_2 A_1}{2(I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2)} + \frac{Pa^2 I_2(-h^3 A_2 + ah^2 A_2 + 3lI_1)}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (45)$$

$$M_C = -\frac{3Pa^2 l^2 I_2 A_1}{2(I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2)} + \frac{Pa^2 I_2(-h^3 A_2 + ah^2 A_2 + 3lI_1)}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (46)$$

$$M_D = -\frac{3Pa^2 l^2 I_2 A_1}{2(I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2)} + \frac{Pa^2[A_2 I_1 l(3h^2 - ah) + I_2(2A_2h^3 - aA_2h^2 + 3I_1 l)]}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (47)$$

For the loaded portal frame of Figure 8, the deformations of the reduced structure due to external loads are

$$d_{10} = -P\left[\frac{a^2 I_1 A_1(3l-2a) + 6hI_2(alA_1 + 2I_1)}{12EI_1 I_2 A_1}\right] \quad (48a)$$

$$d_{20} = \frac{Pah^2}{2EI_1} \quad (48b)$$

$$d_{30} = -\frac{Pa(aI_1 + 2hI_2)}{2EI_1 I_2} \quad (48c)$$

By substituting the values of equations (48a) – (48c) into equations (12)

$$X_1 = \frac{P\left(a^2 I_1 A_1(3l-2a) + 6hI_2(alA_1 + 2I_1)\right)}{I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2} \quad (49a)$$

$$X_2 = \frac{-3Pah^2 I_1 A_2(l-a)}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (49b)$$

$$X_3 = \frac{Pa[h^3 A_2(2aI_1 + hI_2) + 3lI_1(aI_1 + 2hI_2)]}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (49c)$$

Evaluating equation (16) for different points on the structure using the force factors obtained in equations (49a) – (49c)

$$M_A = -Pa + \frac{Pl[a^2 I_1 A_1(3l-2a) + 6hI_2(alA_1 + 2I_1)]}{2(I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2)} + \frac{Pa[A_2 h^3(-aI_1 + hI_2 + 3lI_1) + 3lI_1(aI_1 + 2hI_2)]}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (50)$$

$$M_B = -Pa + \frac{P\left(a^2 I_1 A_1(3l-2a) + 6hI_2(alA_1 + 2I_1)\right)}{2(I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2)} + \frac{Pa[h^3 A_2(2aI_1 + hI_2) + 3lI_1(aI_1 + 2hI_2)]}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (51)$$

$$M_C = -\frac{Pl\left(a^2 I_1 A_1(3l-2a) + 6hI_2(alA_1 + 2I_1)\right)}{2(I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2)} + \frac{Pa[h^3 A_2(2aI_1 + hI_2) + 3lI_1(aI_1 + 2hI_2)]}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (52)$$

$$M_D = -\frac{Pl\left(a^2 I_1 A_1(3l-2a) + 6hI_2(alA_1 + 2I_1)\right)}{2(I_1 A_1 l^3 + 6I_2 A_1 l^2 h + 24hI_1 I_2)} + \frac{Pa[A_2 h^3(-aI_1 + hI_2 + 3lI_1) + 3lI_1(aI_1 + 2hI_2)]}{2(2A_2h^3 lI_1 + 3I_1^2 l^2 + A_2 h^4 I_2 + 6hlI_1 I_2)} \quad (53)$$

IV. DISCUSSION OF RESULTS

The internal stress on the loaded frames is summarized in table 1. The effect of axial deformation is captured by the dimensionless constant s taken as the ratio of the end translational stiffness to the shear stiffness of a member.

$$s_1 = \frac{12EI_1}{h^3} \cdot \frac{h}{EA_1} = \frac{12I_1}{h^2 A_1} \quad (54)$$

$$s_2 = \frac{12EI_2}{l^3} \cdot \frac{l}{EA_2} = \frac{12I_2}{l^2 A_2} \quad (55)$$

When the axial deformation in the columns is ignored $s_1 = 0$ and likewise when axial deformation in the beam is ignored $s_2 = 0$. If axial deformation is ignored in the whole structure, $s_1 = s_2 = 0$.

The internal stress equations enable an easy investigation into the contribution of axial deformation to the internal stresses of statically loaded frames for different kinds of external loads.

For frame 1 (figure 3), the moment at A, M_A is given by equation (19). The contribution of axial deformation in the column ΔM_A is given by

$$\Delta M_A = M_{A(from\ equation\ \ 19)} - M_{A(from\ equation\ \ 19\ when\ s_2=0)}$$

$$= \frac{wl^3 I_1}{12}\left[\frac{4h^3 I_2 + s_2 l^3 I_1}{4h^3 I_2(2lI_1 + hI_1) + s_2 l^3 I_1 (lI_1 + 2hI_2)} - \frac{1}{2lI_1 + hI_2}\right] \quad . \qquad (56)$$

Equation (56) gives the contribution of axial deformation to M_A as a function of h, l, I_1, I_2 and s

By considering the case of a portal frame of length l = 5m, h = 4m. Equation 56 was evaluated to show how the contribution of axial deformation varied with I_1/I_2. The result is shown in Table 2. When plotted on a uniform scale (Figure 9) the relationship between ΔM_B and I_1/I_2 is seen to be linear. This was further justified by a linear regression analysis of the results in Table 2 which was fitted into the model $\Delta M_A = P_1\frac{I_1}{I_2} + P_2$ to obtain $P_1 = -0.004815$ and $P_2 = 0.0001109$ and the fitness parameters sum square of errors (SSE), coefficient of multiple determination (R^2) and the root mean squared error (RMSE) gave 1.099 x 10^{-7}, 0.999952 and 0.0001105 respectively. From Table 2 when $I_1/I_2 = 0$, $\Delta M_A \cong 0$ and when $I_1/I_2 = 10; \Delta M_A = -0.0479$ which represent only a 5% reduction in the calculated bending moment.

In like manner by evaluating the axial contribution in the beam, ΔM_B for varying I_1/I_2 of the portal frame, Table 3 was produced. This was plotted on a uniform scale in Figure 10. From Figure 10 it would be observed that there is also a linear relationship between ΔM_B and I_1/I_2. When fitted into the model $\Delta M_B = P_3\frac{I_1}{I_2} + P_4$ it gave $P_3 = -0.006502$ and $P_4 = -0.0003969$ for the fitness parameters sum square of errors (SSE), coefficient of multiple determination (R^2) and the root mean squared error (RMSE) of 6.51 x 10^{-7}, 0.9999 and 0.000269 respectively.

By pegging I_1/I_2 to a constant value of 0.296 and the variation of ΔM_A with respect to h/L investigated, Table 4 was generated. A detailed plot of Table 4 was presented in Figure 11. From Table 4 when $h/L = 0; \Delta M_A = -3.125$ (about 30% drop in calculated bending moment value) while at $h/L > 0, \Delta M_A$ dropped in magnitude exponentially to values below 1. In like manner, when the variation of ΔM_B with respect to h/L was investigated, Table 5 was generated. A detailed plot of Table 5 was presented in Figure 12. From Table 5 when $h/L = 0; \Delta M_A = -4.1667$ (about 40% drop in calculated bending moment value) while at $h/L > 0, \Delta M_A$ dropped in magnitude exponentially to values below 1.

V. CONCLUSION

The flexibility method was used to simplify the analysis and a summary of the results are presented in table 1. The equations in table 1 would enable an easy evaluation of the internal stresses in loaded rigidly fixed portal frames considering the effect of axial deformation.

From a detailed analysis of frame 1 (Figure 3), it was observed that the contribution of axial deformation is generally very small and can be neglected for reasonable values of I_1/I_2. However, its contribution skyrockets at very low values of h/l i.e. as $h/l => 0$. This depicts the case of an encased single span beam and a complete departure from portal frames under study. This analysis can be extended to the other loaded frames (Figures 4 – 8) using the equations in Table 1. These would enable the determination of safe conditions for ignoring axial deformation under different kinds of loading.

REFERENCES

Ghali A, Neville A. M. (1996) *Structural Analysis: A Unified Classical and Matrix Approach* (3rd Edition) Chapman & Hall London

Graham R, Alan P.(2007). Single Storey Buildings: *Steel Designer's Manual* Sixth Edition, Blackwell Science Ltd, United Kingdom

Hibbeler, R. C.(2006). *Structural Analysis*. Sixth Edition, Pearson Prentice Hall, New Jersey

Leet, K. M., Uang, C.,(2002). *Fundamental of Structural Analysis*. McGraw-Hill ,New York

McGuire, W., Gallagher R. H., Ziemian, R. D.(2000). *Matrix Structural Analysis*,Second Edition, John Wiley & Sons, Inc. New York

Nash, W.,(1998). *Schaum's Outline of Theory and Problems of Strength of Materials*. Fourth Edition, McGraw-Hill Companies, New York

Okonkwo V. O (2012). *Computer-aided Analysis of Multi-storey Steel Frames*. M. Eng. Thesis, Nnamdi Azikiwe University, Awka, Nigeria.

Reynolds, C. E.,and Steedman J. C. (2001). *Reinforced Concrete Designer's Handbook*, 10th Edition) E&FN Spon, Taylor & Francis Group, London

Samuelsson A., and Zienkiewicz O. C.(2006), Review: History of the Stiffness Method. *International Journal for Numerical Methods in Engineering*. Vol. 67: 149 – 157

Stroud K. A.,(1995). *Engineering Mathematics*. Fourth Edition, Macmillan Press Ltd, London.

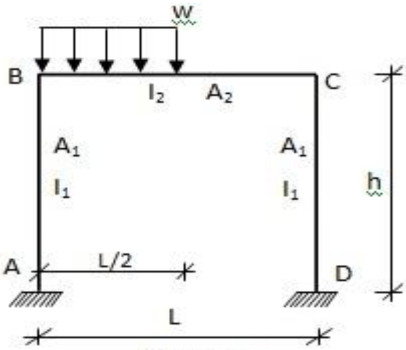


Figure 4

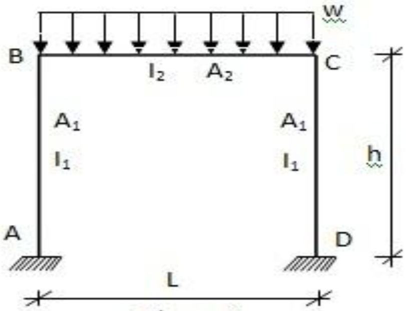


Figure 3

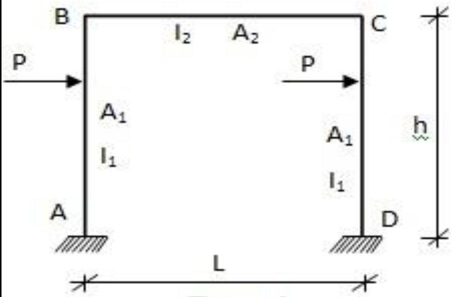


Figure 6

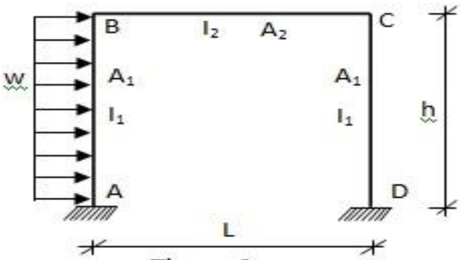


Figure 5

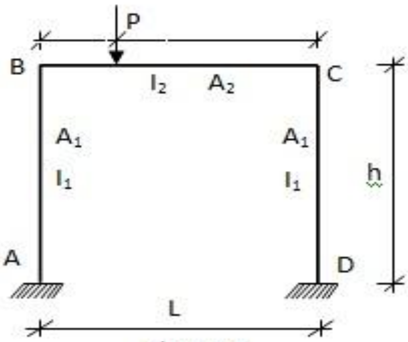


Figure 8

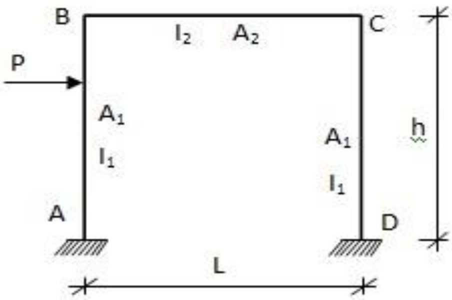
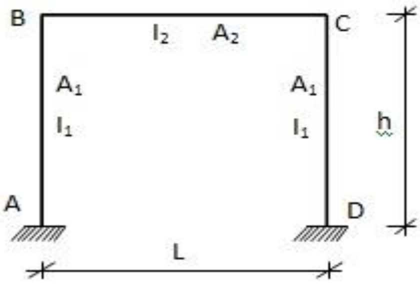


Figure 7

Table 1: Internal stresses for a loaded rigid frame



A_1 = Cross-sectional area of the columns

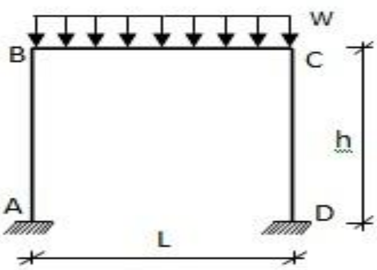
I_1 = Second moment of area of the column cross-section

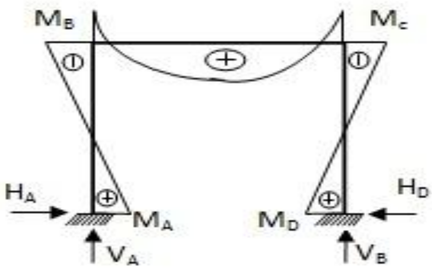
A_2 = Cross-sectional area of the beam

I_2 = Second moment of area of the beam cross-section

S/No LOADED FRAME REMARKS

1



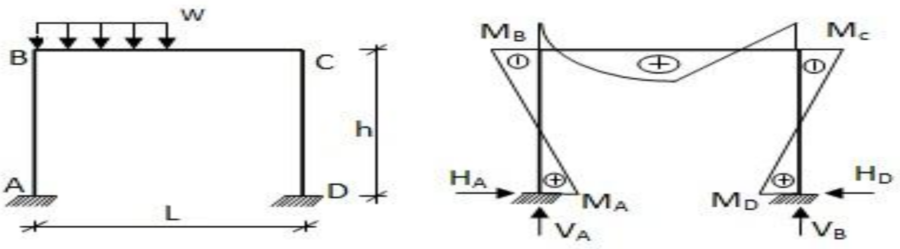


$$M_A = M_D = \frac{wl^3 I_1(4h^3 I_2 - s_2 l^3 I_1)}{12[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_B = M_C = \frac{-wl^3 I_1(8I_2 h^3 + s_2 l^3 I_1)}{12[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$V_A = V_D = \frac{wl}{2} \qquad H_A = H_D = \frac{wh^2 l^3 I_1 I_2}{4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)}$$

See equations (19) – (22)



$$M_A = \frac{wl^2[4h^3 I_2(3hI_2 + 8lI_1) + l^3 I_1(s_1 lI_1 + 6s_2 hI_2)]}{48[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3(lI_1 + 2hI_2)]} - \frac{wl^4(5hI_1 + 24hI_2)}{64(l^2 I_1 + 6hl^2 I_2 + 2s_1 h^3 I_2)}$$

$$M_B = -\frac{wl^4(5hI_1 + 24hI_2)}{64(l^2 I_1 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{wl^2[4h^3 I_2(3hI_2 + 2lI_1) + s_2 l^3 I_1(lI_1 + 6hI_2)]}{48[4h^3 I_2(2lhI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_C = -\frac{wl(3I_1 l^3 + 24I_2 l^2 h + 16s_1 h^3 I_2)}{64(I_1 l^3 + 6I_2 l^2 h + 2s_1 h^3 I_2)} + \frac{wl^2[4h^3 I_2(3hI_2 + 2lI_1) + s_2 l^3 I_1(lI_1 + 6hI_2)]}{48[4h^3 I_2(2lhI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

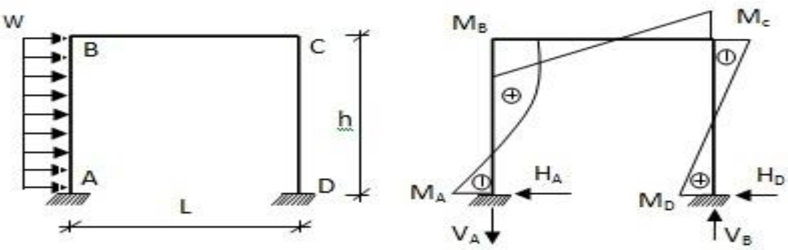
$$M_D = \frac{wl^2[4h^3 I_2(3hI_2 + 8lI_1) + l^3 I_1(s_1 lI_1 + 6s_2 hI_2)]}{48[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3(lI_1 + 2hI_2)]} - \frac{wl(3I_1 l^3 + 24I_2 l^2 h + 16s_1 h^3 I_2)}{64(I_1 l^3 + 6I_2 l^2 h + 2s_1 h^3 I_2)}$$

$$V_D = \frac{wl(3l^3 I_1 + 24hl^2 I_2 + 16s_1 h^3 I_2)}{32(l^3 I_1 + 6hl^2 I_2 + 2s_1 h^3 I_2)} \qquad V_A = \frac{wl}{2} - V_D$$

$$H_A = H_D = \frac{wh^2 l^3 I_1 I_2}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3(lI_1 + 2hI_2)]}$$

See equations (26) – (29)

3



$$M_A = -\frac{wh^2(I_1 l^3 + 5hl^2 I_2 + 2s_2 h^3 I_2)}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{wh^3 I_2(9h^2 lI_1 + 5h^3 I_2 + s_2 l^3 I_1)}{6[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_B = \frac{wh^3 l^2 I_2}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{wh^3 I_2(-4h^3 I_2 + s_2 l^3 I_1)}{6[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_C = -\frac{wh^3 l^2 I_2}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{wh^3 I_2(-4h^3 I_2 + s_2 l^3 I_1)}{6[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

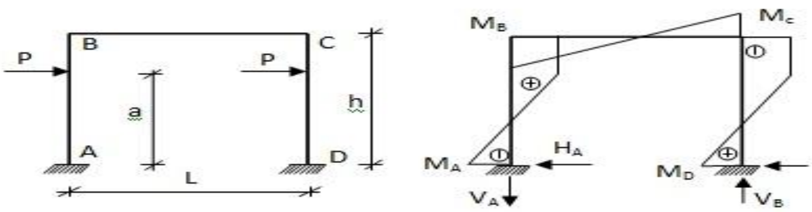
$$M_D = -\frac{wh^3 l^2 I_2}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{wh^3 I_2(9h^2 lI_1 + 5h^3 I_2 + s_2 l^3 I_1)}{6[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$V_A = V_B = \frac{wh^3 lI_2}{I_1 l^3 + 6I_2 l^2 h + 2s_1 I_2 h^3}$$

$$H_D = \frac{wh^4 I_2(3lI_1 + 2hI_2)}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]} \qquad H_A = wh - H_D$$

See equations (32) – (35)

4



$$M_A = -Pa\left(1 - \frac{3al^2 I_2}{I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2}\right) \qquad M_B = \frac{3Pa^2 l^2 I_2}{I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2}$$

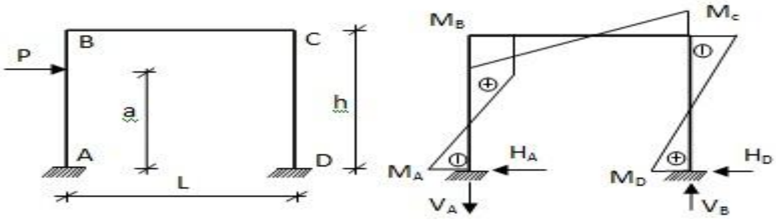
H_D

$$M_C = -\frac{3Pa^2 l^2 I_2 A_1}{A_1 I_1 l^3 + 6A_1 hl^2 I_2 + 24hI_1 I_2} \qquad M_D = Pa\left(1 - \frac{3al^2 I_2}{I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2}\right)$$

$$V_A = V_B = \frac{6Pa^2 lI_2}{l^3 I_1 + 6l^2 hI_2 + 2s_1 h^3 I_2} \qquad H_A = H_B = P$$

See equations (38) – (41)

5



$$M_A = -Pa + \frac{3Pa^2 l^2 I_2}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{Pa^2[4I_1 I_2 hl(3h-a) + I_2(8h^3 I_2 - 4ah^3 I_2 + s_2 l^3 I_1)]}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_B = \frac{3Pa^2 l^2 I_2}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{Pa^2 I_2(-4h^3 I_2 + 4ah^2 I_2 + s_2 l^3 I_1)}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

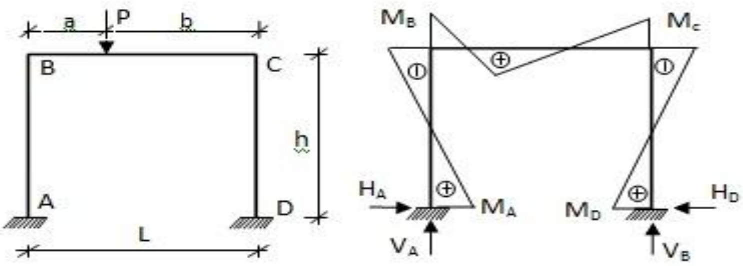
$$M_C = -\frac{3Pa^2 l^2 I_2}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{Pa^2 I_2(-4h^3 I_2 + 4ah^2 I_2 + s_2 l^3 I_1)}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_D = -\frac{3Pa^2 l^2 I_2}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{Pa^2[4I_1 I_2 hl(3h-a) + I_2(8h^3 I_2 - 4ah^3 I_2 + s_2 l^3 I_1)]}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$V_A = V_B = \frac{3Pa^2 lI_2}{l^3 I_1 + 6l^2 hI_2 + 2s_1 h^3 I_2}$$

$$H_D = \frac{2Pa^2 I_2[lI_1(3h-a) + hI_2(3h-2a)]}{4h^3 I_2(2lI_1) + s_2 l^3 I_1(lI_1 + 2hI_2)} \qquad H_A = P - H_D$$

See equations (44) – (47)



$$M_A = -Pa + \frac{Pl\left[a^2 I_1 (3l - 2a) + hI_2\left(6al + s_1 h^2\right)\right]}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{Pa\left[4h^3 I_2(-aI_1 + hI_2 + 3lI_1) + s_2 l^3 I_1(aI_1 + 2hI_2)\right]}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_B = -Pa + \frac{Pl\left[a^2 I_1 (3l - 2a) + hI_2\left(6al + s_1 h^2\right)\right]}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{Pa\left[4h^3 I_2(2aI_1 + hI_2) + s_2 l^3 I_1(aI_1 + 2hI_2)\right]}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_C = -\frac{Pl\left[a^2 I_1 (3l - 2a) + hI_2\left(6al + s_1 h^2\right)\right]}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{Pa\left[4h^3 I_2(2aI_1 + hI_2) + s_2 l^3 I_1(aI_1 + 2hI_2)\right]}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$M_D = -\frac{Pl\left[a^2 I_1 (3l - 2a) + hI_2\left(6al + s_1 h^2\right)\right]}{2(I_1 l^3 + 6hl^2 I_2 + 2s_1 h^3 I_2)} + \frac{Pa\left[4h^3 I_2(-aI_1 + hI_2 + 3lI_1) + s_2 l^3 I_1(aI_1 + 2hI_2)\right]}{2[4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)]}$$

$$V_D = \frac{P\left[a^2 I_1(3l - 2a) + 6ahlI_2 + s_1 h^3 I_2\right]}{l^3 I_1 + 6hl^2 I_2 + 2s_1 h^3 I_2} \qquad V_A = P - V_D$$

$$H_A = H_B = \frac{6Pah^2 I_1 I_2 (l - a)}{4h^3 I_2(2lI_1 + hI_2) + s_2 l^3 I_1(lI_1 + 2hI_2)}$$

See equations (50) – (53)

Table 2: Axial deformation contribution ΔM_A for different values of I_1/I_2

w =1kN/m L = 5m H = 0.4m h = 4m

I_1/I_2	0	1	2	3	4	5
ΔM_A	0	-0.0045	-0.0095	-0.0144	-0.0192	-0.0241
I_1/I_2	6	7	8	9	10	
ΔM_A	-0.0289	-0.0337	-0.0384	-0.0432	-0.0479	

Table 3: Axial deformation contribution ΔM_B for different values of I_1/I_2

w =1kN/m L = 5m H = 0.4m h = 4m

I_1/I_2	0	1	2	3	4	5
ΔM_B	0	-0.0066	-0.0135	-0.0201	-0.0267	-0.0332
I_1/I_2	6	7	8	9	10	
ΔM_B	-0.0396	-0.0460	-0.0524	-0.0587	-0.0651	

Table 4: Axial deformation contribution ΔM_A for different values of h/l

w =1kN/m L = 5m $I_1/I_2 = 0.296$

h/l	0	0.1	0.2	0.3	0.4	0.5
ΔM_A	-3.1250	-0.5409	-0.0823	-0.0239	-0.0096	-0.0047
h/l	0.6	0.7	0.8	0.9	1.0	
ΔM_A	-0.0026	-0.0015	-0.0010	-0.0006	-0.0004	

Table 5: Axial deformation contribution ΔM_B for different values of h/l

w =1kN/m L = 5m $^{I_1}/_{I_2} = 0.296$

$^{h}/_{l}$	0	0.1	0.2	0.3	0.4	0.5
ΔM_A	-4.1667	-0.7668	-0.1208	-0.0359	-0.0146	-0.0072
$^{h}/_{l}$	0.6	0.7	0.8	0.9	1.0	
ΔM_A	-0.0040	-0.0024	-0.0015	-0.0010	-0.0007	

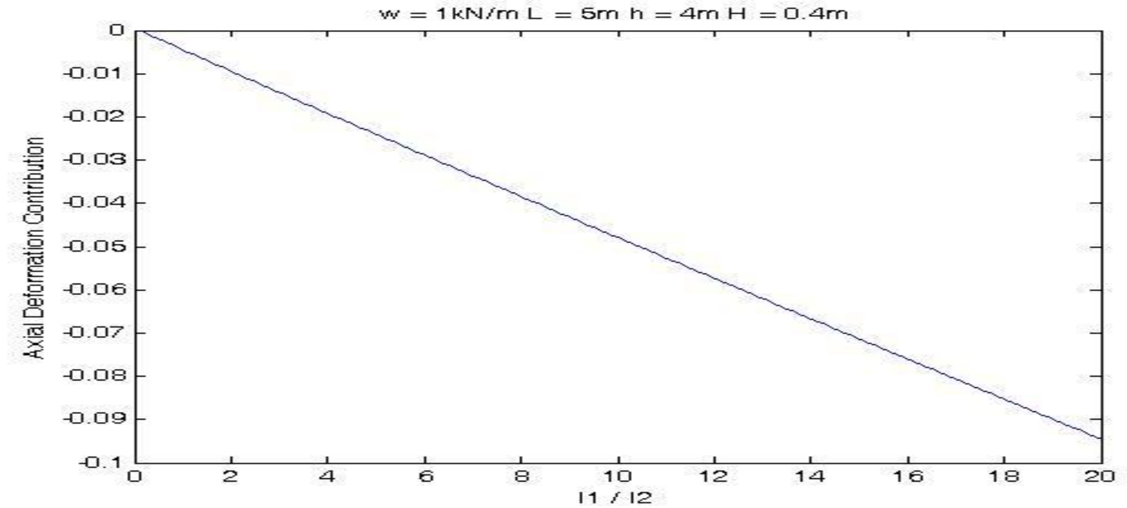


Figure 9: A graph of ΔM_A against $^{I_1}/_{I_2}$

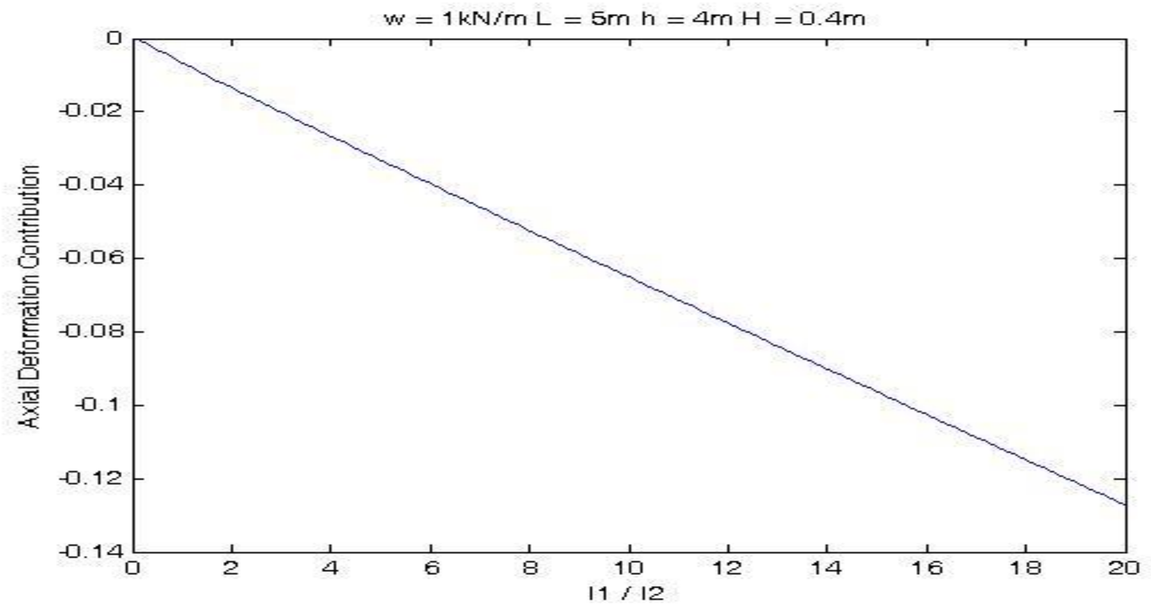


Figure 10: A graph of ΔM_B against $^{I_1}/_{I_2}$

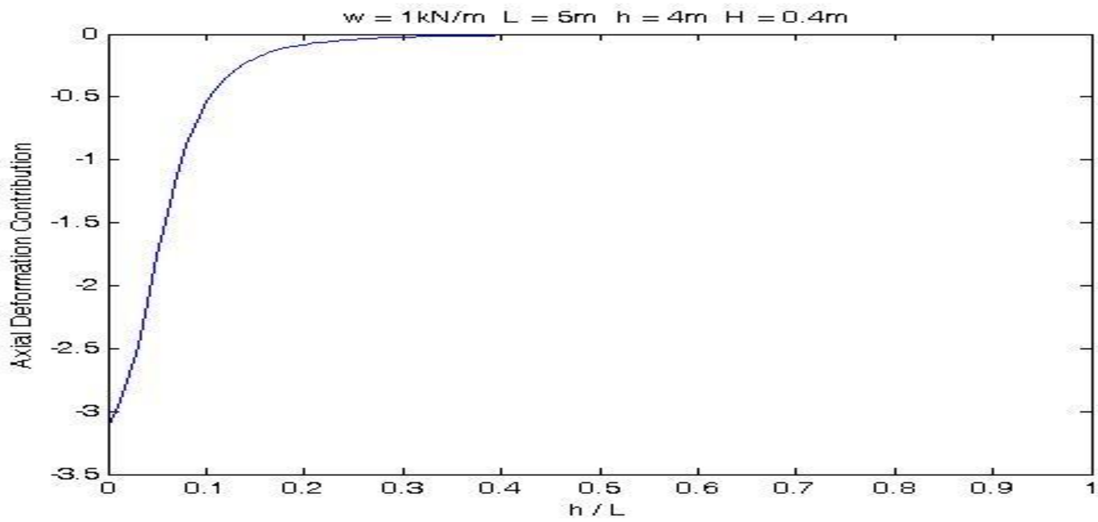


Figure 11: A graph of ΔM_A against $h/_L$

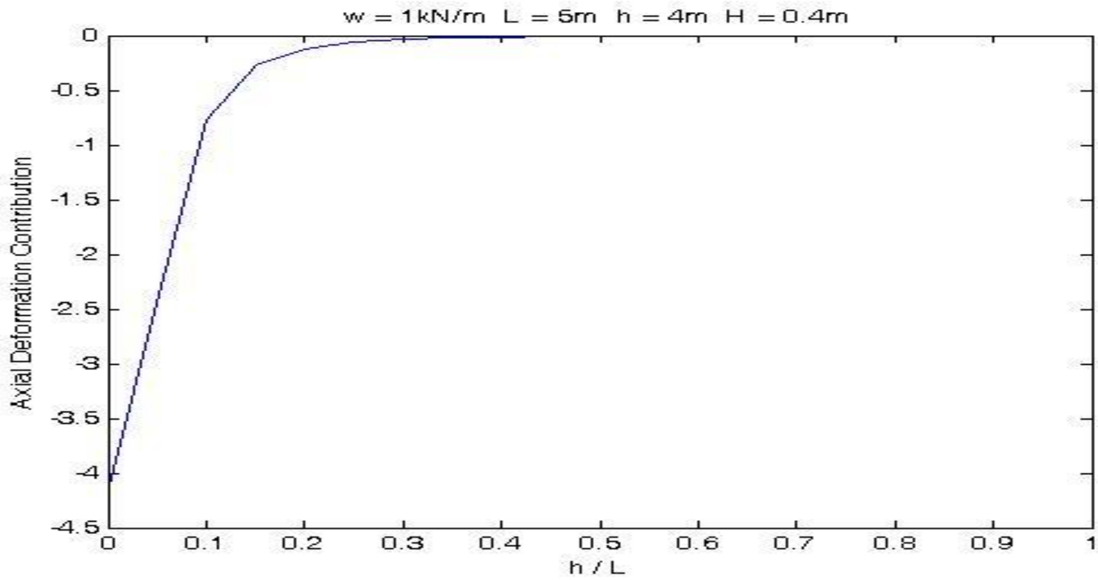


Figure 12: A graph of ΔM_B against $h/_L$