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Modeling Competitive Fleet Management Dynamics Through Perfect Aggregation and Optimal Effort Analysis.

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Abstract

This paper develops a dynamic modeling framework to investigate the evolution of traveler population and fleet management effort. Using a perfect aggregation approach, a time-evolution equation governing the traveler population is derived. Fleet effort is formulated as a function of net economic benefit, defined by the difference between transport revenue and operating costs. Under the assumption of fixed transport costs, an initial system of two ordinary differential equations is reduced to a single aggregated equation. The long-term dynamics of the model are examined through equilibrium and local stability analysis. The value E_{max} is identified as the optimal fleet management effort. The findings offer analytical insights that can support fleet operators in designing strategies that enhance operational efficiency and maximize economic returns.

Key words:Optimal fleet units, perfect aggregation.

1. Introduction

Aggregation is the process of reducing the dimensionality of a dynamical system to facilitate mathematical analysis. A system can be aggregated in two ways: either through approximate aggregation or through perfect aggregation. Approximation aggregation technique involves substituting a complex large scale mathematical model with a simplified aggregated system derived through justified approximations. Such approximations become feasible when certain variables evolve on a much faster time scale than others. Fast-moving variables typically reach equilibrium quickly, allowing them to be replaced with their corresponding non-trivial equilibrium values. When these fast variables converge to an asymptotically stable equilibrium, they are similarly substituted by their equilibrium states. However, if the fast variables exhibit periodic behavior over time, they are instead represented by their time-averaged values. In some situations, aggregation is carried out by disregarding the dynamics of slow variables; though, as noted in [5], this may not give a clear picture when observing the system.

Perfect aggregation approach works by substituting a large-scale mathematical model often containing many variables with a simplified, aggregated representation. The reduced model is constructed by introducing new global variables, typically defined through integrals or combinations of the original systems variables, as noted in [1,4]. A key assumption is that the initial system is conservative. For example, consider a model describing interactions between two prey populations with densities X_1 and X_2 respectively, and a predator population with density Y, given as:

$$\begin{aligned} \dot{X}_1 &= r_1 X_1 (1 - \frac{q_{13} Y}{K_1}) \\ \dot{X}_2 &= r_2 X_2 (1 - \frac{q_{23} Y}{k_2}) \\ \dot{Y} &= -r_3 Y + q_{31} X_1 Y + q_{32} X_2 Y, \end{aligned} \tag{1}$$

where r_1, r_2 and r_3 represent the intrinsic growth rates for the densities X_1, X_2 and Y respectively. q_{13}, q_{23}, q_{31} and q_{32} are parameters in the system which describe the interaction strengths among the species. This system can be aggregated by defining new variables given as:

$$N_1 = X_1 + X_2$$

$$N_2 = Y,$$

which combine the two prey populations into a single compartment. Aggregation is feasible when the parameters in (1) satisfy the following relations:

$$r_1 = r_2 = r$$

$$r_3 = s$$

$$K_1 = K_2 = K$$

$$q_{13} = q_{23} = q$$

$$q_{31} = q_{32} = q'.$$

Under these constraints, the model reduces to:

$$\begin{aligned} \dot{N}_1 &= r N_1 (1 - \frac{q N_2}{k}) \\ \dot{N}_2 &= -s N_2 + q' N_1 N_2. \end{aligned} \tag{2}$$

These parameter restrictions illustrate that perfect aggregation can only occur under specific structural conditions. In large systems, such ideal aggregation is often unattainable because the requirements for combining variables are narrow and depend on particular parameter relationships. Nonetheless, when applicable, the aggregated model is just a condensed prototype of the original model.

2. Public transport outlook

Public transport in Kenya begun during colonial period. In 1934, the overseas company of London brought the local 13 fleet buses on 12 routes. It was latter changed to Kenya bus service and enjoyed monopoly in the transport sector until 1970's when matatu owners association and the county bus owners association stepped in to bridge the gap and meet the demand steepened by rural urban migration, see [3, 8, 10].

Matatu transport industry started as an illegal venture due to stringent government policies but was eventually legalized in 1973 by Kenya's first president Jomo Kenyatta ,see [7]. Soon after legalizing matatu industry in Kenya, it became a blessing to Kenyan economy as they created jobs to many Kenyans. Since matatu operators were hardworking Kenyans devoted to their work,this led to the growth of young republic in terms of economy.

Bodaboda business originated by early 1990's from Uganda through busia town in western part of the country Kenya and spread to the other towns in the country, see [6]. Until now the business has already rooted in all parts of the country. The transport services was a Ugandan initiative that grew from Busia on Kenya-Uganda border. The name bodaboda emerged from from the term 'boarder' otherwise Kenya-Uganda boarder. Businessmen used to transport passengers and goods across Kenya-Uganda roads. Bicycles were majorly used in a flat terrain roads and later the mode of transport shifted to motorcycles which was more convenient, see [9].

From the literature, much has been commented and written on the advantages of fleets to the transport sector, see [11] but little attention has been paid to the profitability of the business and effects of over-investment in fleet units beyond an optimum number. Similarly, there isn't a dynamical model for fleet management as the number of fleet units increases.

3. Fleet management model dynamics

The equation describe the rate of change in travelers number who are being served by fleet units. Given a number of travelers at a given area growing naturally at the rate equivalent to human population growth rate and commuting between two zones then their services are being responded by public fleets through the activity of satisfying travel needs that cannot be met by walking . If b and d denotes the recruitment to traveling class due to birth and removal from traveling class due to death respectively, then we have:

$$F(T) = \dot{T} = bT - dT = rT \tag{3}$$

where $r = b - d$ is the intristic growth rate. The exact solution of $F(T)$ with the initial number $F(0) = T_0$ is

$$T(t) = T_0 e^{rt}. \tag{4}$$

The function (3) will grow exponentially if $r > 0$ or perish if $r < 0$, such function is valid within a short period of time and cannot persist forever due to traveler number limiting factors like fleet units competition. Taking into account the limiting factors, the traditional exponential equation yields;

$$F(T) = \dot{T} = rT(1 - \frac{T}{K}), \tag{5}$$

which mimics a logistic growth equation in ecology and has been employed in several population models, as illustrated in the work of [2]. In equation (5) above, $r > 0$ is the intristic growth rate, $K > 0$ is the carying capacity which is essentially maximum number of travellers a zone can accomodate. When $T < K$, $\dot{T} > 0$ and the travelers number grows. If the term $\frac{T}{K}$ is small, then $F(T) = \dot{T} = rT$ and the travelers number will grow exponentially. Lastly when $T > K$,$\dot{T} < 0$ and the travelers number decline since the population at the respective zones will be reduced by increasing fleet units. The differential equation (5) has two equilibrium points; the trivial $T = 0$ and the non-trivial $T = K$. Any number below or above K will always shift asymtotically to K.

Consider travelers number who are being served by the fleet units on the basis that fleets are allowed to serve travelers in between two zones. Let

T be travelers population number at time t and that $T(t) = T_1(t) + T_2(t)$ where $T_1(t)$ and $T_2(t)$ represent traveler's number at zonal area 1 and 2 respectively in time t. Since the travelers are increasing depending on the usual human population growth, then the rapid population growth assumes a logistic growth equation at each zone and is given as;

$$\begin{aligned} F(T_1) &= \frac{dT_1}{dt} = r_1 T_1 (1 - \frac{T_1}{K_1}) \\ F(T_2) &= \frac{dT_2}{dt} = r_2 T_2 (1 - \frac{T_2}{K_2}), \end{aligned} \tag{6}$$

where $r_1, r_2 > 0$ and $K_1, K_2 > 0$ respectively represent the intrinsic growth rates and the carrying capacities which essentially is the maximum number of travelers a zone can accommodate (can be limited by factors such as fleet gathering effort) at each of the two zones.

Since travelers are being served by the fleet units, then it mimics the concept of predation, the predatorwood function can then be given as G as a function of T travelers number and E fleet management effort. This describes how the fleet units serves the passengers. Furthermore we can have $G(T, E) = (qT)E$ where $q > 0$ is a constant proportionality which denotes the chance of a fleet operator getting a traveler (customer) at the zonal stage point. $q = 1$ is the maximum value in the range $0 \leq q \leq 1$ and may only occur when the fleet waiting time is zero. We also assume $qT = g(T, E)$, where the function $g(T, E)$ is the reaction function which denotes the number of travelers successfully served by fleet units as a function of travelers number. It describe the way the fleet units respond to the changing number of travelers. This functional reaction function depends linearly on travelers number, thus we have;

$$G(T, E) = qTE. \tag{7}$$

Combining equation (6) and (7) yields;

$$F(T_1) = \frac{dT_1}{dt} = r_1T_1(1 - \frac{T_1}{K_1}) - q_1T_1E_1$$
$$F(T_2) = \frac{dT_2}{dt} = r_2T_2(1 - \frac{T_2}{K_2}) - q_2T_2E_2, \quad (8)$$

where $q_1, q_2 > 0$ are the constant proportionalities which denotes the chance of a fleet operator getting a traveler at zone 1 and zone 2 respectively. Considering migration of people, equation (8) forms a complete model with two ODEs governing the previous variables as:

$$F(T_1) = \frac{dT_1}{d\tau} = (mT_2 - m'T_1) + \varepsilon[r_1T_1(1 - \frac{T_1}{K_1}) - q_1T_1E_1]$$
$$F(T_2) = \frac{dT_2}{d\tau} = (m'T_1 - mT_2) + \varepsilon[r_2T_2(1 - \frac{T_2}{K_2}) - q_2T_2E_2], \quad (9)$$

where m and m' are population migration rates within the two zones, $\varepsilon << 1$ is a small dimensionless parameter such that $\varepsilon = \tau t$. Equation (9) above contain two parts: the first one describes migration between two zones at fast time scale $\varepsilon = \tau t$ (relative to the slower time scale t), while the second part describe travellers natural growth and travellers removal rate by fleet units at each zone and holds at a slow time scale t. Setting $\varepsilon = 0$ in eqution (9), asuming that the fast dynamcs attained stable equilibrium state then we obtain only fast dynamics since variables on fast time scale rapidly attain equlibrium and vary when observing the system and is given as;

$$F(T_1) = \frac{dT_1}{d\tau} = (mT_2 - m'T_1)$$
$$F(T_2) = \frac{dT_2}{d\tau} = (m'T_1 - mT_2), \quad (10)$$

where τ denotes the fast time with $\tau = \frac{t}{\varepsilon}$. To achieve fast equllibrium, we perform the following fast calculations on equation (10). When $\frac{dT_1}{d\tau}, \frac{dT_2}{d\tau} = 0$, we have the following calculations;

$$mT_2^* = m'T_1^*$$

yield:

$$T_1^* = \frac{m}{m'}T_2^*$$
$$T_2^* = \frac{m'}{m}T_1^*, \quad (11)$$

where T_1^* and T_2^* are are travelers number at equilibrium. Since $T(t) = T_1(t) + T_2(t)$ or $T_2^* = T - T_1^*$, then T_1^* in the first part of equation (11) can take the form:

$$T_1^* = \frac{m}{m'}(T - T_1^*)$$

or

$$T_1^*(1 + \frac{m}{m'}) = \frac{m}{m'}T$$

or

$$T_1^* = \frac{m}{m+m'}T.$$

Similar calculations on the second part of equation (11) yields:

$$T_2^* = \frac{m'}{m+m'}T.$$

Combining the two gives the followig fast equllibriums

$$\begin{aligned} T_1^* &= \frac{m}{m+m'}T \\ T_2^* &= \frac{m'}{m+m'}T. \end{aligned} \tag{12}$$

Also the fast model (10) is conservative since the total travellers number $T(t) = T_1(t) + T_2(t)$ remain constant. Consequently, system (9) is conservative and that we can perfom perfect agregation to achieve a new system with the variable given as:

$$T = T_1 + T_2,$$

where travelers population at each zone is agregated into a single compartment which is actually total travelers population. Substituting fast equilibrium in equation (11) into equation (9) while considering the fact that $\tau = \frac{t}{\varepsilon}$ yields:

$$\begin{aligned} \frac{\varepsilon dT_1}{dt} &= (m * \frac{m'}{m}T_1 - m'T_1) + \varepsilon[r_1T_1(1 - \frac{T_1}{K_1}) - q_1T_1E_1] \\ \frac{\varepsilon dT_2}{dt} &= (m' * \frac{m}{m'}T_2 - mT_2) + \varepsilon[r_2T_2(1 - \frac{T_2}{K_2}) - q_2T_2E_2], \end{aligned}$$

or

$$\begin{aligned} \frac{dT_1}{dt} &= r_1T_1(1 - \frac{T_1}{K_1}) - q_1T_1E_1 \\ \frac{dT_2}{dt} &= r_2T_2(1 - \frac{T_2}{K_2}) - q_2T_2E_2. \end{aligned} \tag{13}$$

Furthermore, we allow parameters in equation (9) to assume the following set of relations;

$$\begin{aligned} r_1\kappa_1^* + r_2\kappa_2^* &= r \\ \frac{r_1\kappa_1^{*2}}{K_1} + \frac{r_2\kappa_2^{*2}}{K_2} &= \frac{r}{K} \end{aligned}$$

$$q_1\kappa_1^* = q_2\kappa_2^* = q$$
$$E_1 = E_2 = E,$$

considering this and (12), system (13) is agregated into a single compartment that reads:

$$\frac{dT_1}{dt} + \frac{dT_2}{dt} = r_1T_1 + r_2T_2 - (\frac{r_1T_1^2}{K_1} + \frac{r_2T_2^2}{K_2}) - (q_1T_1E_1 + q_2T_2E_2).$$

Let $\kappa_1 = \frac{m}{m+m'}$ and $\kappa_2 = \frac{m'}{m+m'}$ such that equation (12) can take the form

$$T_1^* = \kappa_1^*T$$

$$T_2^* = \kappa_2^*T$$

then,

$$\frac{d\kappa_1^*T}{dt} + \frac{d\kappa_2^*T}{dt} = r_1\kappa_1^*T + r_2\kappa_2^*T - (\frac{r_1\kappa_1^{*2}T^2}{K_1} + \frac{r_2\kappa_2^{*2}T^2}{K_2}) - (q_1\kappa_1^*TE_1 + q_2\kappa_2^*TE_2)$$

or

$$\frac{\kappa_1^*dT}{dt} + \frac{\kappa_2^*dT}{dt} = T(r_1\kappa_1^* + r_2\kappa_2^*) - T^2(\frac{r_1\kappa_1^{*2}}{K_1} + \frac{r_2\kappa_2^{*2}}{K_2}) - TE(q_1\kappa_1^* + q_2\kappa_2^*)$$

or

$$(\kappa_1^* + \kappa_2^*)\frac{dT}{dt} = Tr - T^2\frac{r}{K} - TEq.$$

Since;

$$\kappa_1^* + \kappa_2^* = \frac{m}{m+m'} + \frac{m'}{m+m'} = 1$$

then, we arrive at;

$$\frac{dT}{dt} = rT(1 - \frac{T}{K}) - qTE. \tag{14}$$

Change in the fleet units serving effort evolves depending on the difference between the total benefit and the cost C incurred on operating the fleet, which include the cost of fuel used per unit distance and fleet maintenance like; cost of fleet oil, insurance taxes and salaries. This is summarized as;

$$\dot{E}\alpha(benefit - cost), \tag{15}$$

with benefit being the product of the fleet fare between two zones and the total number of travelers served by the fleet units and is denoted by $P \times G(T, E)$. Total cost is given by the product of fleet consumption per unit effort C and fleet operating effort E or $cost = E \times C$ then we obtain

$$\dot{E} = \beta(P(G(T, E)) - CE). \tag{16}$$

This model assumes that with time, if the benefit from fleet units is larger than the cost incured in the activity, new fleet units are entering into the transport activity. Since $G(T, E) = qTE$, then we have

$$\dot{E} = \beta E(qPT - C), \tag{17}$$

where C is the cost per unit effort. Combining equation (14) and (17) yield:

$$\begin{aligned} \dot{T} &= rT(1 - \frac{T}{K}) - qTE \\ \dot{E} &= \beta E(qPT - C). \end{aligned} \tag{18}$$

4.Optimal effort analysis

Optimal effort is a suitable effort value in the dynamics between travelers number and the fleet units gathering effort. In this section we shall present the dynamics of travelers number growth and fleet gathering effort to obtain the optimal value E_{max} as illustrated in the work of [12]. Allowing β to take the maximum value $\beta = 1$ in the range $0 \leq \beta \leq 1$, the system in equation (18) yields

$$\begin{aligned} \dot{T} &= rT(1 - \frac{T}{K}) - qTE \\ \dot{E} &= E(qPT - C). \end{aligned} \tag{19}$$

At equilibrium, that is when $(\dot{T}, \dot{E} = 0)$, system (19) gives;

$$\begin{aligned} E^* &= \frac{r}{q}(1 - \frac{T^*}{K}) \\ T^* &= \frac{C}{Pq}. \end{aligned} \tag{20}$$

Using the second Equation in the first Equation of system (20) yields sustainable total travellers gathering activities which is given by the expression

$$E^* = \frac{r}{q}(1 - \frac{C}{PKq}). \tag{21}$$

Also from the expression (21) above, we have q as the only parameter depending on fleet gathering effort. Other parameters depends travelers number distribution but not on fleet gatherng effort distribution. Hence optimal sustainable fleet gathering effort takes the form:

$$E^*(q) = \frac{r}{q}(1 - \frac{C}{PKq}). \tag{22}$$

From the logistic model, optimal level is attain when $T^* = \frac{K}{2}$. The second equation of the system (20) can be expresed as:

$$q = \frac{C}{T^*P},$$

by making q the subject of the formula. For optimal chances we have

$$q_{opt} = \frac{C}{T^*P}|_{T^*=\frac{K}{2}}$$

or

$$q_{opt} = \frac{2C}{PK}. \tag{23}$$

Equation (23) above shows that if the chance of fleet operator getting a traveler q_{opt}, fleet units operating cost (C), the fleet fare (P) and the total carying capacity (K) are known, then optimal fleet gathering effort must be chosen. The choice of particular fleet gathering effort allow maximum fleet gathering activities. In this case the optimal fleet gathering effort is given by the relation:

$$E^*(q_{opt}) = \frac{r}{q_{opt}}(1 - \frac{C}{PKq_{opt}}) = \frac{rPK}{4C}. \tag{24}$$

Also, since fleets incur cost per unit travelling effort denoted as C_i where C_i depends on the fuel price and fleet maintainance cost like repair, fleet oil, Insurance taxes and salaries. Thus the term C_i can take the form $C_i = C + n_i$, where C occur at normal fuel price, while n_i is the fuel tax (when $n_i > 0$) and fuel subsidy (when $n_i < 0$).

Making further assumptions, the cost is allowed to have minimal value

$$\forall i \quad 0 \leq C_{min} \leq C_i. \tag{25}$$

Assumption (25) is realistic since the maximum return cannot be realized without minimum cost per unit time. Thus from the bounded domain of cost, the maximum fleet gathering chance becomes:

$$q_{opt} = \frac{2C_{min}}{PK}, \tag{26}$$

and the maximum effort to give maximum returns becomes:

$$maxE^*(q_{opt}) = \frac{r}{q_{opt}}(1 - \frac{C_{min}}{PKq_{opt}}) = \frac{rPK}{4C_{min}}. \tag{27}$$

Which is the maximum sustainable fleet gathering effort which gives optimal profit.

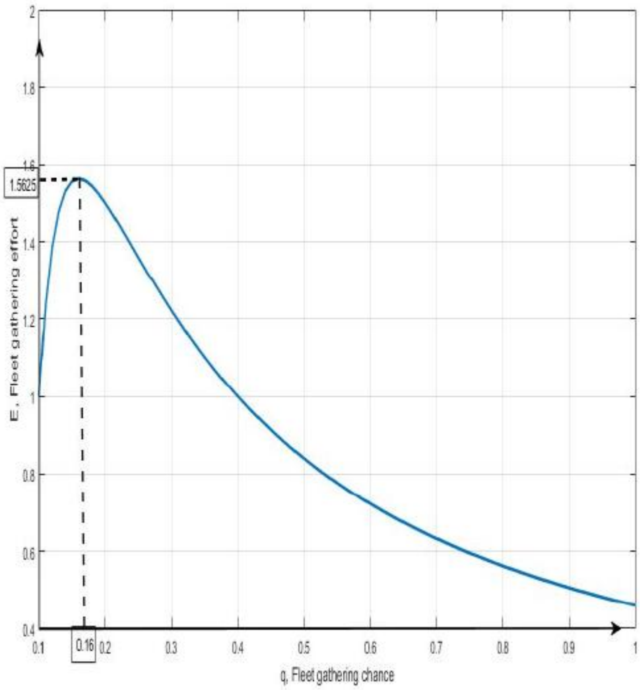

Figure 1: Optimal effort distribution for a fixed cost

The figure (1) present the shape of a function $E^*(q_{opt}) = \frac{r}{q_{opt}}(1 - \frac{C_{min}}{PKq_{opt}})$. We see clearly that there is a maximum value of $E^*(q)$ at $q_{opt} = 0.16$ for a fixed cost C and $q_{opt} = \frac{2C_{min}}{PK}$, where the parameters $P = 2.5, K = 1$ and

$C_{min} = 0.2$. Since E is a variable changing with different values of q, increase in chances of a fleet operator getting a passenger (decrease in fleet operator waiting time) motivates growth of fleet units due to lucrative economic opportunity just before flooding the market and decreasing asymptotically.

5.Conclusion

From this study, a mathematical model was successfully formulated using a system of differential equations to describe the dynamic interactions between the travelers' number and fleet management effort
The models long-term behavior was investigated, offering insights into the system's response to small disturbances.
A key objective of determining the threshold value was successfully achieved. The analysis revealed that there exists an optimal threshold value $E_{max} = \frac{rPK}{4C}$, is the value of fleet engagement beyond which additional fleet units do not significantly enhance service efficiency.

6.Recommendation

Based on the analysis, it is recommended that the government regulate the importation of fleet units in line with human population growth trends. Such regulation would ensure that the number of fleet units remains optimal to meet demand efficiently without creating oversupply.
Further research should examine how pricing strategies interact with optimal fleet deployment under competitive conditions. Approaches such as dynamic pricing, bundled services, or loyalty programs could help operators maximize revenue and retain market share while remaining within optimal resource limitsparticularly during peak demand periods.

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