Resolution of Navier-Stokes equations using mixed finite element method and the (D+N) boundary condition

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ABSTRACT

In this paper we introduced the Navier-Stokes equations with a boundary (D+N) condition. We have shown the existence and uniqueness of the solution of the weak formulation obtained. We used the discretization by mixed finite element method. In order to evaluate the performance of the method, the numerical results are compared with some previously published works or with others coming from commercial code like Adina system.

Keywords- **Navier-Stokes Equations, Finite Element Method, Adina system.**

I. INTRODUCTION

The mixed finite element method, based on the velocity-pressure formulation, is being increasingly used for the numerical solution of the Navier-Stokes equations. In this paper we will discuss the mixed finite element method for the nonlinear Navier-Stokes problem with a boundary condition noted (D+N). Under suitable existence and uniqueness conditions of the weak formulation of this problem.

The plan of the paper is as follows. Section II presents the model problem used in this paper. The discretization by mixed finite elements is described in section III. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in Section IV.

II. INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

We consider the steady-state Navier-stokes equations for the flow;

$$-\nu\nabla^2\vec{u}+\vec{u}.\nabla\vec{u}+\nabla p=\vec{f} \text{ in } \Omega \quad (1)$$

$$\nabla.\vec{u}=0 \text{ in } \Omega \quad (2)$$

Where $\nu \succ 0$ a given constant is called the kinematic viscosity, $\vec{u}$ is the fluid velocity, p is the pressure field. ∇ is the gradient, $\nabla.$ is the divergence and ∇^2 is the Laplacien operator, $\vec{f} \in \left[L^2(\Omega)\right]^2$.

The boundary value problem that is considered is the system (1)-(2) posed on two or three-dimensional domain Ω , with boundary conditions on $\partial\Omega$ noted $(\mathrm{D}+\mathrm{N})$ and given by

$$(\mathrm{D}+\mathrm{N}): \mathrm{b}_0\vec{u}+\nu\frac{\partial\vec{u}}{\partial n}-\vec{n}p=\vec{t} \text{ and } \vec{u}.\vec{n}=l \text{ in } \partial\Omega \quad (3)$$

Ω is a bounded and connected domain of IR^2 with a Lipchitz continuous boundary $\partial\Omega$ where $\vec{n}$ denote the outward pointing normal to the boundary, $\vec{t} \in \left[L^2(\partial\Omega)\right]^2$ *and* b_0 is a function defined on $\partial\Omega$ verify: There are two strictly positive constants α_0 and β_0 such that:

$$\alpha_0 \le b_0(x) \le \beta_0 \text{ for all } x \in \partial\Omega \quad (4)$$

The presence of the nonlinear convection term $\vec{u}.\nabla\vec{u}$ means that boundary value problems associated with the Navier-stokes equations can have more than one solution.

We define the spaces:

$$h^1(\Omega)=\left\{u:\Omega\to IR/u;\frac{\partial\vec{u}}{\partial x};\frac{\partial\vec{u}}{\partial y}\in L^2(\Omega)\right\} \quad (5)$$

$$H^1(\Omega)=\left[h^1(\Omega)\right]^2 \quad (6)$$

$$L_0^2(\Omega)=\left\{q\in L^2(\Omega)/\int_\Omega q\,d\Omega=0\right\} \quad (7)$$

$$H_{n,l}^1(\Omega)=\left\{\vec{v}\in H^1(\Omega)/\vec{v}.\vec{n}=l \text{ in } \partial\Omega\right\} \quad (8)$$

$$V_{n,l}(\Omega)=\left\{\vec{v}\in H_{n,l}{}^1(\Omega)/\nabla.\vec{v}=0 \text{ in } \Omega\right\} \quad (9)$$

$$\tilde{V}_{n,l}(\Omega)=\left\{\vec{v}\in\left[D(\Omega)\right]^2;\vec{v}.\vec{n}=l \text{ in } \partial\Omega \text{ and } \nabla.\vec{v}=0 \text{ in } \Omega\right\} \quad (10)$$

The standard weak formulation of the Navier-Stokes flow problem (1) – (2)-(3) is the following

$find\left(\vec{u},p\right)\in H_{n,l}^1(\Omega)\times L_0^2(\Omega)$ such that

$$\begin{cases} \nu\int_\Omega \nabla\vec{u}:\nabla\vec{v}d\Omega+\int_{\partial\Omega}b_0\vec{u}.\vec{v}d\gamma+\int_\Omega(\vec{u}.\nabla\vec{u})\vec{v}d\Omega-\int_\Omega p\nabla.\vec{v}d\Omega=\int_{\partial\Omega}\vec{t}.\vec{v}d\Omega+\int_\Omega\vec{f}.\vec{v}d\Omega \\ -\int_\Omega q\nabla.\vec{u}d\Omega=0 \end{cases} \quad (11)$$

$for\, all\,(\vec{v},q)\in H^1_{n,l}(\Omega)\times L^2_0(\Omega)$.

Let the bilinear forms $a: H^1_{n,l}(\Omega)\times H^1_{n,l}(\Omega)\to IR;\ b: H^1_{n,l}(\Omega)\times L^2_0(\Omega)\to IR$, and the trilinear forms $a: H^1_{n,l}\times H^1_{n,l}\times H^1_{n,l}\to IR$ and $a_1: H^1_{n,l}\times H^1_{n,l}\times H^1_{n,l}\to IR$

$$a(\vec{u},\vec{v})=\nu\int_\Omega\nabla\vec{u}:\nabla\vec{v}d\Omega+\int_{\partial\Omega}b_0\vec{u}.\vec{v}d\gamma \quad (12)$$

$$b(\vec{u},q)=-\int_\Omega q\nabla.\vec{u}d\Omega \quad (13)$$

$$d(q,\Psi)=\int_\Omega q.\Psi d\Omega \quad (14)$$

$$c(\vec{z},\vec{u},\vec{v})=\int_\Omega(\vec{z}.\nabla\vec{u}).\vec{v}d\Omega \quad (15)$$

Given the functional $L: L^2_0(\Omega)\to IR$

$$L(\vec{v})=\int_{\partial\Omega}\vec{t}.\vec{v}d\Omega+\int_\Omega\vec{f}.\vec{v}d\Omega \quad (16)$$

The underlying weak formulation (11) may be restated as $find\,(\vec{u},p)\in H^1_{n,l}(\Omega)\times L^2_0(\Omega)$ such that

$$\begin{cases} a(\vec{u},\vec{v})+c(\vec{u},\vec{u},\vec{v})+b(\vec{v},q)=L(\vec{v}) \\ b(\vec{u},q)=0 \end{cases} \quad (17)$$

$for\, all\,(\vec{v},q)\in H^1_{n,l}(\Omega)\times L^2_0(\Omega)$.

In the sequel we can assume that $l=0$ and $\vec{t}=\vec{0}$ and we will study the existence and uniqueness of the solution of the problem (17), for that we need the following results.

Lemma 2.1.

1) There is two strictly positive constants c_1 and c_2 such that

$$c_1\|\vec{v}\|_{1,\Omega}\le\|\vec{v}\|_{J,\Omega}\le c_2\|\vec{v}\|_{1,\Omega}\ \ for\, all\,\vec{v}\in H^1_{n,0}(\Omega) \quad (18)$$

With

$$\|\vec{v}\|_{J,\Omega}=\left(|\vec{v}|^2_{1,\Omega}+\|\vec{v}\|^2_{0,\partial\Omega}\right)^{\frac{1}{2}} =\left(\int_\Omega\nabla\vec{v}:\nabla\vec{v}d\Omega+\int_{\partial\Omega}\vec{v}.\vec{v}d\gamma\right)^{\frac{1}{2}} \quad (19)$$

$$\|\vec{v}\|_{1,\Omega}=\left(|\vec{v}|^2_{1,\Omega}+\|\vec{v}\|^2_{0,\Omega}\right)^{\frac{1}{2}} =\left(\int_\Omega\nabla\vec{v}:\nabla\vec{v}d\Omega+\int_\Omega\vec{v}.\vec{v}d\gamma\right)^{\frac{1}{2}} \quad (20)$$

2) $\left(H^1_{n,0}(\Omega),\|.\|_{J,\Omega}\right)$ is a real Hilbert space

3) There exists $M\succ 0$ such that

$$a(\vec{u},\vec{v})\le M\|\vec{u}\|_{J,\Omega}\|\vec{v}\|_{J,\Omega} \quad (21)$$

$for\, all\,(\vec{u},\vec{v})\in H^1_{n,0}(\Omega)\times H^1_{n,0}(\Omega)$

4) $b(\vec{v},q)\le\sqrt{2}\|q\|_{0,\Omega}\|\vec{v}\|_{J,\Omega}$ (22)

$for\, all\,(\vec{v},q)\in H^1_{n,0}(\Omega)\times L^2_0(\Omega)$

5) There exists $m\succ 0$ *such* that:

$$c(\vec{z},\vec{u},\vec{v})\le m\|\vec{z}\|_{J,\Omega}.\|\vec{u}\|_{J,\Omega}.\|\vec{v}\|_{J,\Omega} \quad (23)$$

$for\, all\ \vec{z},\vec{u},\vec{v}\in H^1_{n,0}(\Omega)$

6) $c(\vec{z},\vec{u},\vec{v})=-c(\vec{z},\vec{v},\vec{u})\ for\ all\ \vec{z},\vec{u},\vec{v}\in V_{n,0}(\Omega)$ (24)

7) $c(\vec{z},\vec{u},\vec{u})=0\ for\ all\ \vec{z},\vec{u},\in V_{n,0}(\Omega)$ (25)

8) $a_1(\vec{w},\vec{v},\vec{v})\ge\alpha\|\vec{v}\|^2_{J,\Omega}\ and\ a(\vec{v},\vec{v})\ge\alpha\|\vec{v}\|^2_{J,\Omega}$ (26)

$for\ all\ \vec{w},\vec{v},\in V_{n,0}(\Omega)$

9) $\vec{u}_m\to\vec{u}\ \ weakly\, in\, V_{n,0}(\Omega)\ (as\ m\to\infty)\ implies\ that$

$$\lim_{m\to\infty}a_1(\vec{u}_m,\vec{u}_m,\vec{v})=a_1(\vec{u},\vec{u},\vec{v})\ \ for\,\vec{v}\in V_{n,0}(\Omega) \quad (27)$$

Proof. For 1, 2, 3 and 4 see [9].

5) Let $\vec{z},\vec{u},\vec{v}\in H^1_{n,0}(\Omega)$, we have

$$|c(\vec{z},\vec{u},\vec{v})|\le\beta'\|\vec{z}\|_{1,\Omega}.|u|_{1,\Omega}.\|\vec{v}\|_{1,\Omega},$$

(see lemma 2.1, chapter IV in [6]). (18), (19) and (20) implies

$$|c(\vec{z},\vec{u},\vec{v})|\le\frac{\beta'}{c_1}\|\vec{z}\|_{J,\Omega}.\|\vec{u}\|_{J,\Omega}.\|\vec{v}\|_{J,\Omega},$$

6) Let $(\vec{z},\vec{u},\vec{v})\in(V_{n,0}(\Omega))^3$, we have

$$c(\vec{z},\vec{u},\vec{v})+c(\vec{z},\vec{v},\vec{u})=\int_\Omega\vec{z}.(\nabla\vec{u}.\vec{v}+\nabla\vec{v}.\vec{u})d\Omega =\int_\Omega\vec{z}.\nabla(\vec{u}.\vec{v})d\Omega$$

By Green formula we have

$$c(\vec{z},\vec{u},\vec{v})+c(\vec{z},\vec{v},\vec{u})=\int_{\partial\Omega}(\vec{z}.\vec{n})(\vec{u}.\vec{v})d\gamma-\int_\Omega div\,\vec{z}(\vec{u}.\vec{v})d\Omega$$

Since $\vec{z}\in V_{n,0}(\Omega)$, then $\vec{z}.\vec{n}=0$ and $\mathrm{div}\,\vec{z}=\vec{0}$ therefore

$$c(\vec{z},\vec{u},\vec{v})=-c(\vec{z},\vec{v},\vec{u})$$

7) It is easy from 6.

8) Let $\vec{w}\in V_{n,0}(\Omega)$, using (24) and (4) then gives then gives

$$\begin{aligned} a_1(\vec{w},\vec{v},\vec{v}) &= a(\vec{v},\vec{v})+c(\vec{w},\vec{v},\vec{v}) \\ &= a(\vec{v},\vec{v}) \\ &= \nu\int_\Omega \nabla\vec{v}:\nabla\vec{v}d\Omega+\int_{\partial\Omega} b_0\vec{v}.\vec{v}d\gamma \\ &\le \nu\int_\Omega \nabla\vec{v}:\nabla\vec{v}d\Omega+\alpha_0\int_{\partial\Omega} \vec{v}.\vec{v}d\gamma \\ &\le \alpha(\int_\Omega \nabla\vec{v}:\nabla\vec{v}d\Omega+\alpha_0\int_{\partial\Omega} \vec{v}.\vec{v}d\gamma) \\ &= \alpha\|\vec{v}\|^2_{J,\Omega} \end{aligned}$$

with $\alpha=\max(\alpha_0,\nu)$.

9) The same proof of V.Girault and P.A. Raviart in [6] page 115.

Theorem2.2. b is satisfies the inf-sup condition:

There exists a constant $\beta \succ 0$ such that

$$\sup_{\vec{v}\in H^1_{n,0}} \frac{\int_\Omega q\nabla.\vec{v}d\Omega}{\|\vec{v}\|_{J,\Omega}} \ge \beta\|q\|_{0,\Omega} \text{ for all } q\in L^2_{0,\Omega}(\Omega) \qquad (28)$$

Proof. Let $q\in L^2_{0,\Omega}(\Omega)$, We have

$$\sup_{\vec{v}\in H^1_0(\Omega)} \frac{\int_\Omega q\nabla.\vec{v}d\Omega}{|\vec{v}|_{1,\Omega}} \ge \beta\|q\|_{0,\Omega}$$ [6], since

$$H^1_0(\Omega)=\left\{\vec{v}\in H^1(\Omega)/\vec{v}=\vec{0}\ in\ \partial\Omega\right\}\subset H^1_{n,0}(\Omega)$$

and $\|\vec{v}\|_{J,\Omega}=|\vec{v}|_{1,\Omega}$ *if* $\vec{v}\in H^1_0(\Omega)$, then

$$\begin{aligned} \sup_{\vec{v}\in H^1_{n,0}} \frac{\int_\Omega q\nabla.\vec{v}d\Omega}{\|\vec{v}\|_{J,\Omega}} &\ge \sup_{\vec{v}\in H^1_0(\Omega)} \frac{\int_\Omega q\nabla.\vec{v}d\Omega}{\|\vec{v}\|_{J,\Omega}} \\ &= \sup_{\vec{v}\in H^1_0(\Omega)} \frac{\int_\Omega q\nabla.\vec{v}d\Omega}{|\vec{v}|_{1,\Omega}} \\ &\ge \beta\|q\|_{0,\Omega} \end{aligned}$$

We define the "big" symmetric bilinear form

$$B((\vec{u},p);(\vec{v},q))=a(\vec{u},\vec{v})+b(\vec{v},p)+b(\vec{u},q) \qquad (29)$$

According the theorems 1.2 and 1.4, chapter IV in [6], (26) and (27) ensure the existence at least one pair $(\vec{u},p)\in H^1_{n,l}(\Omega)\times L^2_0(\Omega)$ satisfies (17).

We define

$$\mathrm{N}=\sup_{\vec{z},\vec{u},\vec{v}\in H^1_{n,0}(\Omega)} \frac{|c(\vec{z},\vec{u},\vec{v})|}{\|\vec{z}\|_{J,\Omega}.\|\vec{u}\|_{J,\Omega}.\|\vec{v}\|_{J,\Omega}} \qquad (30)$$

$$\|\vec{f}\|_* = \sup_{\vec{v}\in H^1_{n,0}} \frac{\int_\Omega \vec{f}.\vec{v}d\Omega}{\|\vec{v}\|_{J,\Omega}} \qquad (31)$$

Then a well-know (sufficient) condition for uniqueness is that forcing function is small in the sense that

$$\|\vec{f}\|_* \le \frac{\nu^2}{N}$$

(it suffices to apply theorems 1.3 and 1.4 chapter IV in [6]).

Theorem 2.3.

Assume that ν and $\vec{f}\in\left(L^2(\Omega)\right)^2 \le \frac{\nu^2}{N}$ satisfy the following condition

$$\left|\int_\Omega \vec{f}.\vec{v}d\Omega\right| \le \delta\frac{\nu^2}{N}\|\vec{v}\|_{J,\Omega}\ for\ all\ \vec{v}\in H^1_{n,0}(\Omega) \qquad (32)$$

For some fixed number $\delta\in[0,1]$ then there exists a unique solution $(\vec{u},p)\in H^1_{n,l}(\Omega)\times L^2_0(\Omega)$ of (17) and it holds

$$\|\vec{u}\| \le \delta\frac{\nu}{N} \qquad (33)$$

Proof. The some proof of theorem 2.4 chapter IV in [6].

Lemma 2.4. There are two strictly positive constants s_1 and s_2 such that:

$$s_1\|\vec{v}\|^2_{J,\Omega} \le a(\vec{v},\vec{v}) \le s_2\|\vec{v}\|^2_{J,\Omega} \qquad (34)$$

$$for\ all\ \vec{v}\in H^1_{n,0}(\Omega)$$

Proof. Using (4) and (12) gives,

$$a(\vec{v},\vec{v})=\nu\int_\Omega \nabla\vec{v}:\nabla\vec{v}d\Omega+\int_{\partial\Omega} b_0\vec{v}.\vec{v}d\gamma$$

$$\begin{aligned} &\nu\int_\Omega \nabla\vec{v}:\nabla\vec{v}d\Omega+\alpha_0\int_{\partial\Omega}\vec{v}.\vec{v}d\gamma \le a(\vec{v},\vec{v}) \\ &and\ a(\vec{v},\vec{v}) \le \nu\int_\Omega \nabla\vec{v}:\nabla\vec{v}d\Omega+\beta_0\int_{\partial\Omega}\vec{v}.\vec{v}d\gamma \end{aligned} \qquad (35)$$

We take $s_1=Min(\nu,\alpha_0)$ and $s_2=Max(\nu,\beta_0)$ we obtain (34).

III. MIXED FINITE ELEMENT APPROXIMATION

Our goal here is to consider the stationary Navier-Stokes equations with boundary condition (D+N) in a two-dimensional domain and to approximate then by mixed finite element method.

Mixed finite element discretization of the weak formulation of Navier-stokes equations gives rise to nonlinear system of algebraic equations.

Two classical iterative procedures for solving this system are Newton iteration and Picard iteration.

Let $T_h; h \succ 0,$ be a family of triangulations of Ω. For any $T \in T_h, \omega_T$, is of triangles sharing at least one edge with element T, $\tilde{\omega}_T$ is the set of triangles sharing at least one vertex with T. Also, for an element edge E, ω_E denotes the union of triangles sharing E, while $\tilde{\omega}_T$ is the set of triangles sharing at least one vertex whit E.
Next, ∂T is the set of the tree edges of T we denote by $\varepsilon(T)$ and N_T the set of its edges and vertices, respectively.
We let $\varepsilon_h = \underset{T \in T_h}{\cup} \varepsilon(T)$ denotes the set of all edges split into interior and boundary edges.

$$\varepsilon_h = \varepsilon_{h,\Omega} \cup \varepsilon_{h,\partial\Omega}$$

Where, $\varepsilon_{h,\Omega} = \{E \in \varepsilon_h : E \subset \Omega\}$

$$\varepsilon_{h,\partial\Omega} = \{E \in \varepsilon_h : E \subset \partial\Omega\}$$

We denote by h_T the diameter of a simplex, by h_T the diameter of a face E of T, and we set $h = \max_{T \in T_h} \{h_T\}$.

A discrete weak formulation is defined using finite dimensional spaces $X_{n,0}^h \subset H_{n,0}^1(\Omega)$ and $M_0^h \subset L_0^2(\Omega)$.

The discrete version of (11) is:

$$find\,(\vec{u}_h, p_h) \in X_{n,0}^1(\Omega) \times M_0^h(\Omega) \text{ such that}$$

$$\begin{cases} a(\vec{u}_h, \vec{v}_h) + c(\vec{u}_h, \vec{u}_h, \vec{v}_h) + b(\vec{v}_h, q_h) = L(\vec{v}_h) \\ b(\vec{u}_h, q_h) = 0 \end{cases} \quad (36)$$

$$for\, all\,(\vec{v}_h, q_h) \in X_{n,0}^1(\Omega) \times M_0^h(\Omega) \text{ such that}$$

We define the appropriate bases for the finite element spaces, leading to non linear system of algebraic equations. Linearization of this system using Newton iteration gives the finite dimensional system:

$$find\,(\delta\vec{u}_h, \delta p_h) \in X_{n,0}^1(\Omega) \times M_0^h(\Omega) \text{ such that}$$

$$\begin{cases} c(\delta\vec{u}_h, \vec{u}_h, \vec{v}_h) + c(\vec{u}_h, \delta\vec{u}_h, \vec{v}_h) + \nu \int_\Omega \nabla \delta\vec{u}_h : \nabla \vec{v}_h d\Omega + \\ \qquad \int_{\partial\Omega} b_0 \delta\vec{u}_h . \vec{v}_h d\gamma - \int_\Omega \delta p_h \nabla . \vec{v}_h d\Omega = R_k(\vec{v}_h) \\ -\int_\Omega q_h \nabla . \delta\vec{u}_h d\Omega = r_k(q_h) \end{cases} \quad (37)$$

$$for\, all\,(\vec{v}_h, q_h) \in X_{n,0}^1(\Omega) \times M_0^h(\Omega).$$

Here, $R_k(\vec{v}_h)$ and $r_k(q_h)$ are the non linear residuals associated with the discrete formulations (36).To define the corresponding linear algebra problem, we use a set of vector-valued basis functions. $\{\vec{\varphi}_j\}_{j=1\ldots n_u}$, so that

$$\vec{u}_h = \sum_{j=1}^{n_u} u_j \vec{\varphi}_j;\ \delta\vec{u}_h = \sum_{j=1}^{n_u} \Delta u_j \vec{\varphi}_j \quad (38)$$

We introduce a set of pressure basis functions $\{\Psi_k\}_{k=1\ldots n p}$

and set $p_h = \sum_{j=1}^{n_p} p_k \Psi_k;\ \delta p_h = \sum_{k=1}^{n_p} \Delta p_k \psi_k \quad (39)$

Where n_u and n_p are the numbers of velocity and pressure basis functions, respectively.
We find that the discrete formulation (38) can be expressed as a system of linear equations

$$\begin{pmatrix} A+N+W & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad (40)$$

The system is referred to as the discrete Newton problem. The matrix A is the vector Laplacian matrix and B is the divergence matrix

$$A = [a_{i,j}];$$

$$a_{i,j} = \nu \int_\Omega \nabla \varphi_i : \nabla \varphi_j d\Omega + \int_{\partial\Omega} b_0 \varphi_i . \varphi_i d\gamma \quad (41)$$

$$B = [b_{k,j}];\ b_{k,j} = -\int_\Omega \psi_k : \nabla \varphi_j d\Omega \quad (42)$$

for $i, j = 1,\ldots, n_u$ and $k = 1,\ldots, n_p$.

The vector-convection matrix N and the Newton derivative matrix W are given by

$$N = [n_{i,j}];\ n_{i,j} = \int_\Omega (\vec{u}_h . \nabla \vec{\varphi}_j) \vec{\varphi}_i d\Omega \quad (43)$$

$$W = [w_{i,j}];\ w_{i,j} = \int_\Omega (\vec{\varphi}_j . \nabla \vec{u}_h) \vec{\varphi}_i d\Omega \quad (44)$$

For i and $j = 1,\ldots, n_u$. The Newton derivative matrix is symmetric.
The right-hand side vectors in (41) are

$$f = [f_i];\ f_i = \int_\Omega \vec{f} . \vec{\varphi}_i d\Omega \quad (45)$$

for $j = 1,\ldots, n_u$.

For Picard iteration, we give the discrete problem

$$\begin{pmatrix} A+N & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad (46)$$

IV. NUMERICAL SIMULATION

In this section some numerical results of calculations with mixed finite element Method and ADINA System will be presented. Using our solver, we run the Backward-facing step problem (17) with a number of different model parameters.

Example. Backward-facing step problem.

This example represents flow in a rectangular duct with a sudden expansion. A Poiseuille flow profile is imposed on the inflow boundary $\Gamma_1 = (x = -1, 0 \le y \le 1)$, and a no-flow (zero velocity) condition is imposed on the walls. The (D+N) condition (47) is applied at the outflow boundary $\Gamma_5 = (x = 5, -1 \le y \le 1)$,
and automatically sets the mean outflow pressure to zero.

$$\begin{cases} p - \nu \dfrac{\partial u_x}{\partial x} = 10^{-14} u_x \\ -\dfrac{\partial u_y}{\partial x} = 10^{-14} u_y \end{cases} \qquad (47)$$

With these data, see that the (D+N) condition is satisfied, just take $b_0 = 10^{-14}$ on $\Gamma_5 = (x = 5, -1 \le y \le 1)$;
$b_0 = 10^{28}$ on $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$; $\vec{t} = (10^{28};0)$ on Γ_1;
$\vec{t} = (0;0)$ on the other four boundary

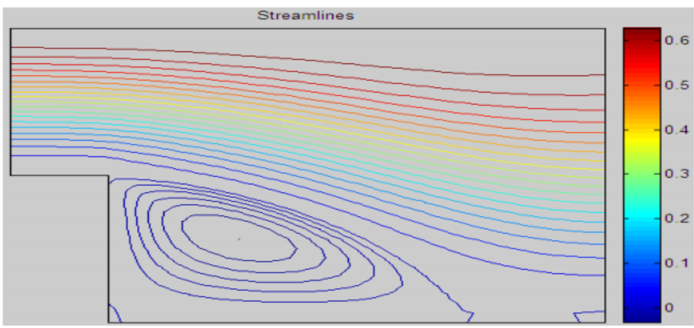

Fig.1. Equally distributed streamline plot associated with a 32×96 square grid, $P_1 - P_2$ approximation and Re=200.

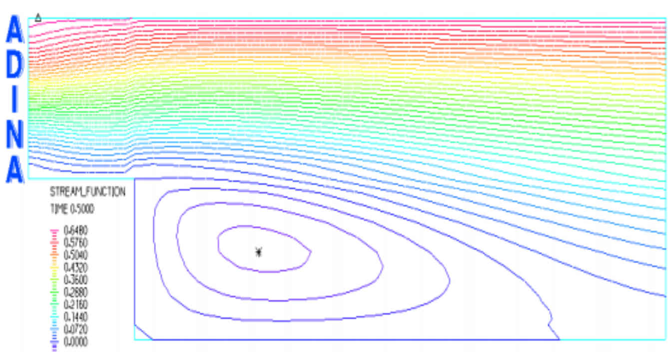

Fig.2. The solution computed with ADINA System. The show the Streamlines associated with a 32×96 square grid.

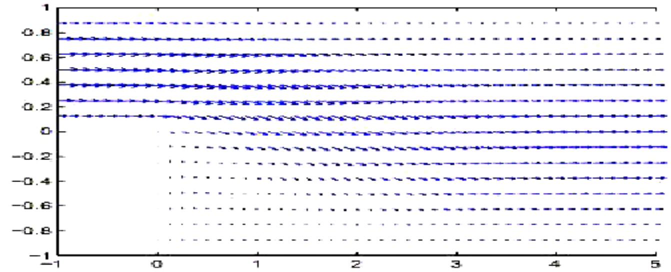

Fig .3. Velocity vectors solution by MFE associated with a 32×96 square grid and Re=200.

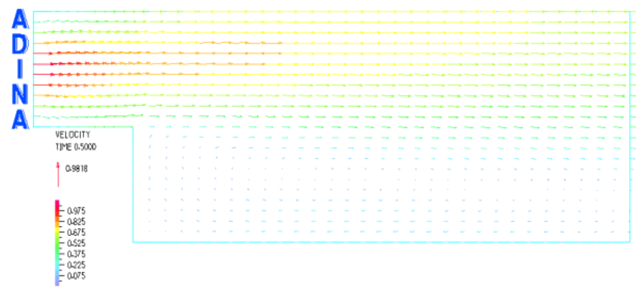

Fig.4. Velocity vectors solution by ADINA system associated with a 32×96 square grid and Re=200.

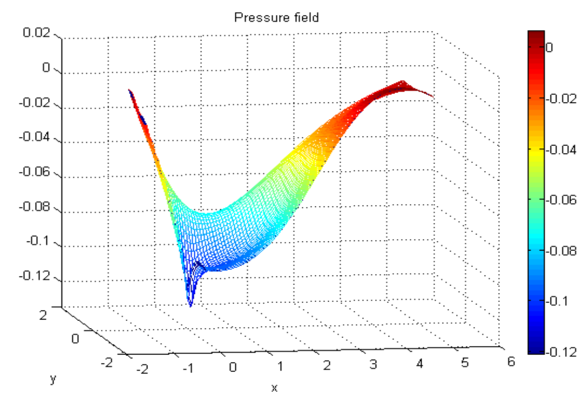

Fig.5. Pressure plot for the flow with a 32×96 square grid.

The two solutions are therefore essentially identical. This is very good indication that our solver is implemented correctly.

V. CONCLUSION

In this work, we were interested in the numerical solution of the partial differential equations by simulating the flow of an incompressible fluid. We applied the mixed finite element method to the resolution of the Navier-Stokes equations with boundary condition noted (D+N).
Numerical results, either resulting from the literature, or resulting from calculation with commercial software like Adina system.

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