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Reproducing Kernel Hilbert Space of A Set Indexed Brownian Motion

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ABSTRACT: *This study researches a representation of set indexed Brownian motion* $X = \{X_A : A \in \mathbf{A}\}$ *via orthonormal basis, based on reproducing kernel Hilbert space (RKHS). The RKHS associated with the set indexed Brownian motion* X *is a Hilbert space of real-valued functions on* T *that is naturally isometric to* $L^2(\mathbf{A})$. *The isometry between these Hilbert spaces leads to useful spectral representations of the set indexed Brownian motion, notably the Karhunen-Loève (KL) representation:* $X_A = \sum e_n E[X_A e_n]$ *where* $\{e_n\}$ *is an orthonormal sequence of centered Gaussian variables.*
In addition, we present two special cases of a representation of a set indexed Brownian motion, when $\mathbf{A} = \mathbf{A}([0,1]^d)$ *and* $\mathbf{A} = \mathbf{A}(Ls)$.

Keywords: *Brownian motion, orthonormal basis, Hilbert Space, Karhunen-Loève.*

I. INTRODUCTION

In this article, we present the representation of set indexed Brownian motion $X = \{X_A : A \in \mathbf{A}\}$ via orthonormal basis, based on reproducing kernel Hilbert space (RKHS). Set indexed Brownian motion is a natural generalization of planar Brownian motion where $\mathbf{A}$ is a collection of compact subsets of a fixed topological space (T,τ). The frame of a set-indexed Brownian motion is not only a new stepto generalize theclassical Brownian motion, but it was proven asa new look upon a Brownian motion (see [Yo09], [Yo15], [MeYo], [He], [IvMe], [Kh], [MeNu]).

RKHS is a robust tooland can be used in a wide variety of areas such as curve fitting,signal analysis and processing, function estimation and model description, differential equations, probability, statistics, nonlinear Burgers equations, empirical risk minimization, fractals, machine learning and etc. (see [Par67], [Par],[Ha],[Sc], [Va], [ScSm], [Dan], [Be], [Cu], [Ge], [Ad]).

Let's assume we havea set indexed Brownian motion on topological and separable space T, with a continuous covariance kernel $R : \mathbf{A} \times \mathbf{A} \to \Re$. We can associate a Hilbert space, which is the reproducing kernel Hilbert space of real-valued functions on T that is naturally isometric to $L^2(\mathbf{A})$.The isometry between these Hilbert spaces leads to useful spectral representations of the set indexed Brownian motion, notably the Karhunen-Loève (KL) theorem.(The KL theorem is a representation of a stochastic process as an infinite linear combination of orthogonal functions, analogous to a Fourier series representation of a function on a bounded interval).In this work, the KL representation of a set indexed Brownian motion is:

$$X_A = \sum e_i E[X_A e_i]$$

Where $\{e_n\}_{n=1}^{\infty}$ an orthonormal sequences of centered Gaussian variables. In addition, in this study we present two special casesof a KL representationof a set indexed Brownian motion, when:

(a) $T = [0,1]^d$ and $\mathbf{A} = \mathbf{A}([0,1]^d) = \{[0,x] : x \in [0,1]^d\}$

(b) $T = [0,1]^d$ and $\mathbf{A} = \mathbf{A}(Ls)$

In the first case, the KL representation of a set indexed Brownian motion is:

$$X_A = X_{[0,x_1]\times[0,x_2]\times\ldots\times[0,x_d]} = (\tfrac{2\sqrt{2}}{\pi})^d \sum_{n=1}^{\infty} \tfrac{1}{(2n+1)^d} e_n \prod_{i=1}^{d} \sin(\tfrac{1}{2}(2n+1)\pi x_i)$$

In the second case:

$$X_A = X_{\cap_m g_m(A)} = X_{\cap_m g_m([0,\frac{k_1}{2^m}]\times[0,\frac{k_2}{2^m}]\times\ldots\times[0,\frac{k_d}{2^m}])} = \lim_{m\to\infty} (\tfrac{2\sqrt{2}}{\pi})^d \sum_{n=1}^{\infty} \tfrac{1}{(2n+1)^d} e_n \prod_{i=1}^{d} \sin(\tfrac{1}{2}(2n+1)\pi \tfrac{k_i}{2^m}) .$$

II. PRELIMINARIES

As in earlier set-indexed works (see [IvMe]), processes and filtrations will be indexed by a class $\mathbf{A}$ whose elements are compact subsets of a fixed σ-compact metric space T. In addition, we assume $\mathbf{A}$ satisfies several natural conditions. We use the definition and notation from [IvMe] and all this section comefrom there.

Definition 1. Let (T,τ) be a non-void sigma-compact connected topologicalspace. A nonempty class $\mathbf{A}$ of compact, connected subsets of T is called an indexed collection if it satisfies the following:

1. $\emptyset \in \mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ s.t. $T = \bigcup_{n=1}^{\infty} B_n^{\circ}$.
2. $\mathbf{A}$ is closed under arbitrary intersections and if $A, B \in \mathbf{A}$ are nonempty, then $A \bigcap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$ and if there exists n such that $A_i \subseteq B_n$ for every i, then $\overline{\bigcup_i A_i} \in \mathbf{A}$.
3. $\sigma(\mathbf{A}) = \mathbf{B}$ where $\mathbf{B}$ is the collection of Borel sets of T.
4. Separability from above: There exist an increasing sequence of finite sub-classes $\mathbf{A}_n = \{A_1^n, \ldots, A_{k_n}^n\} \subseteq \mathbf{A}$ closed under intersection with $\emptyset, B_n \in \mathbf{A}_n(\mathbf{u})$ ($\mathbf{A}_n(\mathbf{u})$ is the class of union of sets in $\mathbf{A}_n$), and a sequence of functions $g_n : \mathbf{A} \to \mathbf{A}_n(\mathbf{u}) \cup T$ such that:

(i) g_n preserves arbitrary intersections and finite unions.

(ii) For each $A \in \mathbf{A}$, $A \subseteq g_n(A)^{\circ}$ and $A = \cap_n g_n(A)$, $g_n(A) \subseteq g_m(A)$ if $n \geq m$.

(iii) $g_n(A) \cap A' \in \mathbf{A}$ if $A, A' \in \mathbf{A}$ and $g_n(A) \cap A' \in \mathbf{A}_n$ if $A \in \mathbf{A}$ and $A' \in \mathbf{A}_n$.

(iv) $g_n(\emptyset) = \emptyset$ for all n.

Examples.

a. The classical example is $T = \Re_+^d$ and $\mathbf{A} = \mathbf{A}(\Re_+^d) = \{[0,x] : x \in \Re_+^d\}$ (or $T = [0,1]^d$ and $\mathbf{A} = \mathbf{A}([0,1]^d)$). This example can be extended to $T = \Re^d$ and $\mathbf{A} = \mathbf{A}(\Re^d) = \{[0,x] : x \in \Re^d\}$, which will give rise to a sort of 2^d-sides process.
b. The example (a) may be generalized as follows. Let $T = \Re_+^d$ or $T = \Re^d$ or $T = [0,1]^d$ and take $\mathbf{A}$ to be the class of compact *lower sets*, i.e. the class of compact subsets A of T satisfying $t \in A$ implies $[0,t] \in \mathbf{A}$. (We denote the class of compact *lower sets* by $\mathbf{A}(Ls)$).
c. Additional examples have been given when T is a "continuous" rooted tree (see [Sl]) and T a subspace of the Skorokhod space, $D[0,1]$ (see [IvMe]).

We will require other classes of sets generated by $\mathbf{A}$. The first is $\mathbf{A}(\mathbf{u})$, which is the class of finite unions of sets in $\mathbf{A}$. We note that $\mathbf{A}(\mathbf{u})$ is itself a lattice with the partial order induced by set inclusion. Let $\mathbf{C}$ consists of all the subsets of T of the form

$C = A \setminus B, A \in \mathbf{A}, B \in \mathbf{A}(\mathbf{u})$.

Any $\mathbf{A}$-indexed function which has a (finitely) additive extension to $\mathbf{C}$ will be called additive (and is easily seen to be additive on $\mathbf{C}(\mathbf{u})$ as well). For stochastic processes, we do not necessarily require that each sample path be additive, but additivity will be imposed in an almost suresense:

A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if ithas an (almost sure) additive extension to $\mathbf{C} : X_\varnothing = 0$ and if $C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \bigcup C_2$ and $C_1 \bigcap C_2 = \varnothing$ then almost surely $X_C = X_{C_1} + X_{C_2}$. In particular, if $C \in \mathbf{C}$ and $C = A \setminus \bigcup_{i=1}^n A_i$, $A, A_1, ..., A_n \in \mathbf{A}$ then almost surely

$$X_C = X_A - \sum_{i=1}^n X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - ... + (-1)^n X_{A \cap \bigcap_{i=1}^n A_i}.$$

We shall always assume that our stochastic processes are additive. We note that a process with an (almost sure) additive extension to $\mathbf{C}$ also has an (almost sure) additive extension to $\mathbf{C(u)}$.

Definition 2. A positive measure σ on $(T, \mathbf{B})$ is called strictly monotone on $\mathbf{A}$ if: $\sigma_{\varnothing'} = 0$ and $\sigma_A < \sigma_B$ for all $A \subset B$, $A, B \in \mathbf{A}$. The collection of these measures is denoted by $M(\mathbf{A})$. ($\varnothing' = \bigcap_{A \in \mathbf{A}, A \neq \varnothing} A$, note that $\varnothing' \neq \varnothing$)

The classical examples for definition is Lebesgue measure or Radon measure when $T = \Re_+^d$ and $\mathbf{A} = \mathbf{A}(\Re_+^d) = \{[0, x] : x \in \Re_+^d\}$

Definition 3. Let $\sigma \in M(\mathbf{A})$. We say that the $\mathbf{A}$-indexed process X is a Brownian motion with variance σ if X can be extended to a finitely additive process on $\mathbf{C(u)}$ and if for disjoint sets $C_1, ..., C_n \in \mathbf{C}$, $X_{C_1}, ..., X_{C_n}$ are independent mean-zero Gaussian random variables with variances $\sigma_{C_1}, ..., \sigma_{C_n}$, respectively. (For any $\sigma \in M(\mathbf{A})$, there exists a set-indexed Brownian motion with variance σ [IvMe]).

III. RKHS OF A SET INDEXED BROWNIAN MOTION

$L^2(\mathbf{A})$ is a separable space,and thusit must have a countable orthonormal basis. Then every set indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ has the representation $X_A = \sum_{i=1}^{\infty} a_i \xi_i$, $a_i \in \Re$ where ξ_i are random variables. Our purpose is to find the representation forthe set indexed Brownian motion via orthonormal basis.

We restart with the RKHS of a set indexed Brownian motion. A RKHSis a Hilbert Space of functions. It can be thought of as a space containing smoother function than the general Hilbert space.

Let $A \in \mathbf{A}$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set indexed Brownian motion. We define the function

$$R(A, \cdot) : \mathbf{A} \to \Re \text{ by } R(A, \cdot) = Cov(X_A, X_.) = E[X_A X_.].$$

Now, we define the set $\Lambda = \{ f : \mathbf{A} \to \Re : f(\cdot) = \sum_{i=1}^n a_i R(A_i, \cdot),\ A_i \in \mathbf{A}, a_i \in \Re, n \geq 1\}$. Define an inner product on Λ by:

$$\langle f, g \rangle = \sum_{i=1}^n \sum_{j=1}^m a_i b_j R(A_i, B_j) \text{ when } f(\cdot) = \sum_{i=1}^n a_i R(A_i, \cdot),\ g(\cdot) = \sum_{j=1}^m b_j R(B_i, \cdot) \quad (1)$$

The fact that R is non-negative definite implies $\langle f, f \rangle = \sum_{i=1}^n a_i^2 R(A_i, A_i) \geq 0$ for all $f \in \Lambda$.From the inner product, we get the following property:

$$f(B) = \sum_{i=1}^n a_i R(A_i, B) = \langle \sum_{i=1}^n a_i R(A_i, \cdot), R(B, \cdot) \rangle = \langle f, R(B, \cdot) \rangle \quad (2)$$

Then for all $f \in \Lambda, B \in \mathbf{A}$, $|f(B)|^2 = |\langle f, R(B, \cdot) \rangle|^2 \leq \langle f, f \rangle \langle R(B, \cdot), R(B, \cdot) \rangle$. The inequality being merely the Schwartz inequality for semi-inner products, which holds as long as $\langle f, f \rangle \geq 0$. Thus, if $\langle f, f \rangle = 0$ then (2) implies that $f = 0$ for all $B \in \mathbf{A}$. Consequently, (1) defines a proper inner product on Λ, and so we thus obtain a norm:

$$\| f \| = \sqrt{\langle f, f \rangle}$$

For $\{ f_n \}_{n=1}^{\infty} \in \Lambda$ we have:

$$| f_n(B) - f_m(B) |^2 = | \langle f_n - f_m, R(B,\cdot) \rangle |^2 \leq \| f_n(B) - f_m(B) \|^2 \| R(B,\cdot) \|^2 \leq$$

$$\leq \| f_n(B) - f_m(B) \|^2 R(B,B)$$

Thusitfollows that if $\{ f_n \}_{n=1}^{\infty}$ is Cauchy in $\| \cdot \|$ then it is a pointwise Cauchy. The closure of Λ denoted by H. Since T is separable and R continuous then H is also separable.Since H is a separable Hilbert space, it must have a countable orthonormal basis.

Define $H(X)$, the so-called "linear part" of the $L^2(\mathbf{A})$ space of the set indexed stochastic process X, as the closure in $L^2(\mathbf{A})$ of

$$\left\{ \sum_{i=1}^{n} a_i X_{A_i},\ A_i \in \mathbf{A}, a_i \in \Re, n \geq 1 \right\},$$

thinned out by identifying all elements indistinguishable in $L^2(\mathbf{A})$ (In other word, elements x, y for which $E[(x-y)^2] = 0$). This contains all distinguishable random variables, with finite variance, obtainable as linear combination of values of the process. There is a linear, one-one mapping $\mathbf{T} : \Lambda \to L^2(\mathbf{A})$ defined by:

$$\mathbf{T}(f) = \mathbf{T}\left(\sum_{i=1}^{n} a_i R(A_i, \cdot) \right) = \sum_{i=1}^{n} a_i X_{A_i}$$

Note that $\mathbf{T}$ is clearly norm preserving and so extends to all H with range equal to all of $H(X)$. (In other words, there is exists $\mathbf{Tr}$ such that $\mathbf{Tr}_{|\Lambda} = \mathbf{T}$, $Dom(\mathbf{Tr}) = H$ and $\mathrm{Im}(\mathbf{Tr}) = H(X)$).

Since H is separable, we now know that $H(X)$ isalso separable. We can use this to build an orthonormal basis for $H(X)$. If $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis for H then $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis for $H(X)$ when $e_n = \mathbf{T}(\phi_n)$. X is a set indexed Brownian motion therefore $E[e_n] = 0$ for all n and

$$X_A = \sum_{i=1}^{\infty} e_i E[X_A e_i] \text{ almost surely,}$$

where the series converges in $L^2(\mathbf{A})$. Easy to see that $\{e_n\}_{n=1}^{\infty}$ an orthonormal sequences of centered Gaussian variables. Since $\mathbf{T}$ was an isometry, it follows that

$$E[X_A e_i] = \langle R(A,\cdot), \phi_i \rangle = \phi_i(A) \text{ for all } i$$

Then $X_A = \sum_{i=1}^{\infty} e_i \phi_i(A)$ almost surely.

RKHS and the KL representation of a set indexed Brownian motion on $\mathbf{A}([0,1]^d)$ and on $\mathbf{A} = \mathbf{A}(Ls)$

Now, we willpresent two special cases of a RKHS and the KL representation of a set indexed Brownian motion, when:

i. $T = [0,1]^d$ and $\mathbf{A}([0,1]^d) = \{[0,x] : x \in [0,1]^d\}$ (see Examples after Definition 1).
ii. $T = [0,1]^d$ and $\mathbf{A} = \mathbf{A}(Ls)$ (see Examples afterDefinition 1).

<u>**First case:**</u> $T = [0,1]^d$ and $\mathbf{A} = \mathbf{A}([0,1]^d) = \{[0,x] : x \in [0,1]^d\}$

Let $X = \{X_A : A \in \mathbf{A}\}$ be a set indexed Brownian motion when $T = [0,1]^d$ and $\mathbf{A} = \mathbf{A}([0,1]^d) = \{[0,x] : x \in [0,1]^d\}$.

Let $A=[0,a_1]\times[0,a_2]\times\ldots\times[0,a_d]\in\mathbf{A}, B=[0,b_1]\times[0,b_2]\times\ldots\times[0,b_d]\in\mathbf{A}$ then

$$R(A,B)=E[X_AX_B]=\min\{a_1,b_1\}\min\{a_2,b_2\}\cdots\min\{a_d,b_d\}$$

Values in set Λ are:

$$f(A)=\sum_{i=1}^{n}a_iR(A_i,A)=\sum_{i=1}^{n}\alpha_i\min\{a_1(i),a_1\}\min\{a_2(i),a_2\}\cdots\min\{a_d(i),a_d\}$$

$$g(A)=\sum_{j=1}^{n}\beta_jR(B_j,A)=\sum_{i=1}^{n}\beta_j\min\{b_1(j),a_1\}\min\{b_2(j),a_2\}\cdots\min\{b_d(j),a_d\}$$

When

$A_i=[0,a_1(i)]\times[0,a_2(i)]\times\ldots\times[0,a_d(i)]\in\mathbf{A}$ and $B_j=[0,b_1(j)]\times[0,b_2(j)]\times\ldots\times[0,b_d(j)]\in\mathbf{A}$

The inner product on Λ is:

$$\langle f,g\rangle=\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_jR(A_i,B_j)=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j\min\{a_1(i),b_1(j)\}\min\{a_2(i),b_2(j)\}\cdots\min\{a_d(i),b_d(j)\}$$

but

$$\min\{a_k(i),b_k(j)\}=\int_0^1 I_{[0,a_k(i)]}(x)I_{[0,b_k(j)]}(x)dx \text{ for } k=1,2,\ldots,d$$

When I_D is the indicator function of $D\in\mathbf{A}$ ($I_D(x)=\begin{cases}1 & ,x\in D\\ 0 & ,x\notin D\end{cases}$).

Therefore, we can rewrite the above as follows:

$$\langle f,g\rangle=\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j\int_0^1 I_{[0,a_1(i)]}(x_1)I_{[0,b_1(j)]}(x_1)dx_1\cdots\int_0^1 I_{[0,a_d(i)]}(x_d)I_{[0,b_d(j)]}(x_d)dx_d=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j\int_{[0,1]^d} I_{[0,a_1(i)]}(x_1)I_{[0,b_1(j)]}(x_1)\cdots I_{[0,a_d(i)]}(x_d)I_{[0,b_d(j)]}(x_d)dx_1\cdots dx_d=$$

$$=\int_{[0,1]^d}\sum_{i=1}^{n}\alpha_i[I_{[0,a_1(i)]}(x_1)\cdots I_{[0,a_d(i)]}(x_d)]\sum_{j=1}^{m}\beta_j[I_{[0,b_1(j)]}(x_1)\cdots I_{[0,b_d(j)]}(x_d)]dx_1\cdots dx_d=$$

$$=\int_{[0,1]^d}f^I(x_1,\ldots,x_n)g^I(x_1,\ldots,x_n)dx_1\cdots dx_d=\int_{[0,1]^d}f^Ig^IdS$$

When

$$f^I(x_1,x_2,\ldots,x_d)=\sum_{i=1}^{n}\alpha_i[I_{[0,a_1(i)]}(x_1)\cdots I_{[0,a_d(i)]}(x_d)$$

$$g^I(x_1,x_2,\ldots,x_d)=\sum_{j=1}^{m}\beta_j[I_{[0,b_1(j)]}(x_1)\cdots I_{[0,b_d(j)]}(x_d)$$

Finally, $\langle f,g\rangle=\int_{[0,1]^d}f^Ig^IdS$.

We define $\Lambda^I=\{f:\mathbf{A}\to\Re:\ f(\cdot)=\int_{\cdot} f^I dS,\ \int_{[0,1]^d}(f^I)^2 dS<\infty\}$ then $R(A,\cdot)\in\Lambda^I$ and $f(A)=\langle f,R(A,\cdot)\rangle=\int_{[0,1]^d} f^I(x_1,...,x_d)I_{[0,a_1]}(x_1)\cdots I_{[0,a_d]}(x_d)dx_1\cdots dx_d$ then $\Lambda^I=\Lambda$. The mapping $\mathbf{T}:\Lambda\to L^2(\mathbf{A})$ is $\mathbf{T}(f)=\mathbf{T}\left(\int_{\cdot} f^I g^I dS\right)=\sum_{i=1}^{n}\alpha_i X_{A_i}$ and from that we get in the same way

$$X_A=\sum_{i=1}^{\infty}e_i E[X_A e_i]\ \text{almost surely.}$$

Karhunen-Loève expansion of a set indexed Brownian motion:

Let $\lambda_1,\lambda_2,...,\lambda_d$ and $\psi_1,\psi_2,...,\psi_d$ be a eigenvalues and normalized eigenfunctions of operator $\Psi:L^2(\mathbf{A})\to L^2(\mathbf{A})$

$$\Psi(\psi(A))=\Psi(\psi(x_1,x_2,...,x_d))=\int_{[0,1]^d}R(A,S)\psi(S)dS=$$

$$=\int_{[0,1]^d}\overbrace{\min\{s_1,x_1\}\min\{s_2,x_2\}\cdots\min\{s_d,x_d\}}^{R(A,S)}\psi(s_1,s_2,...,s_d)ds_1ds_2\cdots ds_d$$

When $A=[0,x_1]\times[0,x_2]\times...\times[0,x_d]\in\mathbf{A}$, $S=[0,s_1]\times[0,s_2]\times...\times[0,s_d]\in\mathbf{A}$.

That is λ_i and ψ_i solve the integral equation

$$\lambda\psi(x_1,x_2,...,x_d)=\int_{[0,1]^d}\min\{s_1,x_1\}\min\{s_2,x_2\}\cdots\min\{s_d,x_d\}\psi(s_1,s_2,...,s_d)ds_1ds_2\cdots ds_d$$

And

$$\int_{[0,1]^d}\psi_i(s_1,s_2,...,s_d)\psi_j(s_1,s_2,...,s_d)ds_1ds_2\cdots ds_d=\begin{cases}1 & ,i=j\\ 0 & ,i\neq j\end{cases}$$

Suppose that, there exist a $\{\zeta_i\}_{i=1}^d$ such that $\psi(x_1,x_2,...,x_d)=\zeta_1(x_1)\zeta_2(x_2)\cdots\zeta_d(x_d)$ for all $(x_1,x_2,...,x_d)\in[0,1]^d$ then

$$\lambda\zeta_1(x_1)\cdots\zeta_d(x_d)=\int_0^1\zeta_1(s_1)\min\{s_1,x_1\}ds_1\cdots\int_0^1\zeta_d(s_d)\min\{s_d,x_d\}ds_d \qquad (3)$$

Denote $\lambda=\mu^d$. If for all i, $\mu\zeta_i(x_i)=\int_0^1\zeta_i(s_i)\min\{s_i,x_i\}ds_i$ then we get the required on (3).It is clearthat, $\mu\zeta_i(x_i)=\int_0^1\zeta_i(s_i)\min\{s_i,x_i\}ds_i=\int_0^{x_i}s_i ds_i+x_i\int_{x_i}^1\zeta_i(s_i)ds_i$ for all i. Differentiating both sides with respect to x_i generates: $\mu\zeta_i'(x_i)=\int_{x_i}^1\zeta_i(s_i)ds_i$, $\mu\zeta_i''(x_i)=-\zeta_i(x_i)$ for all i, togetherwith boundary condition $\zeta_i(0)=0$. The solutions of this pair of differential equations are given by:

$$\zeta_{i,n}(x_i)=\sqrt{2}\sin(\tfrac{1}{2}(2n+1)\pi x_i),\quad \mu_n=\left(\tfrac{2}{(2n+1)\pi}\right)^2$$

Then

$$\psi_n(x_1,x_2,...,x_d)=\zeta_{1,n}(x_1)\zeta_{2,n}(x_2)\cdots\zeta_{d,n}(x_d)=\prod_{i=1}^d\sqrt{2}\sin(\tfrac{1}{2}(2n+1)\pi x_i),\ \lambda_n=\left(\tfrac{2}{(2n+1)\pi}\right)^{2d}$$

The Karhunel-Loève expansion of X is obtained by setting $\phi_i=\sqrt{\lambda_i}\psi_i$, we get:

$$X_A = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \psi_n(A)$$

In other words, $X_{[0,x_1]\times[0,x_2]\times\ldots\times[0,x_d]} = (\frac{2\sqrt{2}}{\pi})^d \sum_{n=1}^{\infty} \frac{1}{(2n+1)^d} e_n \prod_{i=1}^{d} \sin(\frac{1}{2}(2n+1)\pi x_i)$ (4)

Second case: $T = [0,1]^d$ and $\mathbf{A} = \mathbf{A}(Ls)$

Let $x = (x_1, x_2, ..., x_d) \in [0,1]^d$, wedenote $[0,x] = [0,x_1]\times[0,x_2]\times\ldots\times[0,x_d]$. There existsan increasing sequence of finite sub-classes $\mathbf{A}^m([0,1]^d) = \{A_1^m, ..., A_{k_m}^m\} \subseteq \mathbf{A}$ closed under intersection: $\mathbf{A}^m([0,1]^d) = \{[0,x] : x_i = \frac{k_i}{2^m},\ 0 \le k_i \le 2^m,\ i = 1,2,...,d\} \cup \varnothing$ .From the Definition 1 we derive that: $g_m([0,x]) = [0, (\frac{k_1}{2^m}, ..., \frac{k_d}{2^m})] = [0, \frac{k_1}{2^m}]\times[0,\frac{k_2}{2^m}]\times\ldots\times[0,\frac{k_d}{2^m}]$, $g_m(\varnothing) = \varnothing$ and for all $A \in \mathbf{A}(Ls)$, $A = \cap_m g_m(A)$.

Let $A \in \mathbf{A}(Ls)$ then from (4),

$$X_{g_m([0,x_1]\times[0,x_2]\times\ldots\times[0,x_d])} = X_{[0,\frac{k_1}{2^m}]\times[0,\frac{k_2}{2^m}]\times\ldots\times[0,\frac{k_d}{2^m}]} = (\frac{2\sqrt{2}}{\pi})^d \sum_{n=1}^{\infty} \frac{1}{(2n+1)^d} e_n \prod_{i=1}^{d} \sin(\frac{1}{2}(2n+1)\pi \frac{k_i}{2^m})$$

The process X is continuous and then from Definition 1 we derive:

$$\lim_{m\to\infty} X_{g_m([0,x_1]\times[0,x_2]\times\ldots\times[0,x_d])} = \lim_{m\to\infty} (\frac{2\sqrt{2}}{\pi})^d \sum_{n=1}^{\infty} \frac{1}{(2n+1)^d} e_n \prod_{i=1}^{d} \sin(\frac{1}{2}(2n+1)\pi \frac{k_i}{2^m})$$

$$X_A = X_{\cap_m g_m(A)} = X_{\cap_m g_m([0,\frac{k_1}{2^m}]\times[0,\frac{k_2}{2^m}]\times\ldots\times[0,\frac{k_d}{2^m}])} = \lim_{m\to\infty} (\frac{2\sqrt{2}}{\pi})^d \sum_{n=1}^{\infty} \frac{1}{(2n+1)^d} e_n \prod_{i=1}^{d} \sin(\frac{1}{2}(2n+1)\pi \frac{k_i}{2^m})$$

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