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The Queue M/M/1 with Additional Servers for a Longer Queue

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Abstract: *This paper deals with the queuing system M/M/1 with additional servers for a longer queue. Clearly the traffic intensity for this system will depend on the number of additional servers. The expected number of customers in the system, the probability of the additional of one server and the probability of the additional of two servers are obtained under the assumption that the number of additional servers depends on the number of customers in the system. The condition under which the M/M/1 queuing system with additional servers is profitable is discussed. A MATLAB program is used to illustrate this condition numerically. Finally, the maximum likelihood estimators of the parameters for this queuing system are obtained.*

Keywords: *Queuing system, traffic intensity, additional servers, customers, maximum likelihood estimate, queuing length.*

I. Introduction

In many queuing situations, normally the number of servers is dependent on the queue length. For instance, in a bank more and more windows are opened for service when the queues in front of the already open windows get too long. This procedure is sometimes used even in the case of airlines, buses, etc. The basic advantage in such a system is the inherent flexibility of service. Here we shall consider a simplified system of this type.

Suppose we deal with a queuing system in which the arrivals are individually in a Poisson process with parameter λ, and service times of customers are negative exponential with mean μ^{-1}. The first customer to come is the first to be served. As long as there are enough customers for service. As long as the number of customers in the system is greater than or equal to zero and less than or equal to N, there is only one server in the system. As the number of customers in the system increases to more than N and is still less than or equal to 2N, an additional server is added. This additional server is removed when the number of customers in the system decreases to N or less. As soon as the number of customers in the system goes beyond 2N, the number of servers will be three. Similarly, the third server will be taken off when the number of customers falls to 2N or below. Let λ_n and μ_n be the arrival and service rates, respectively, when there are n customers in the system and so, for the described queuing model, we have

$$\lambda_n = \lambda \qquad \text{for every } n \geq 0 ,$$

$$\mu_n = \begin{cases} \mu ; & \text{if } 0 < n \leq N , \\ 2\mu ; & \text{if } N < n \leq 2N, \\ 3\mu ; & \text{if } 2N < n , \end{cases}$$

where, λ, μ and $\rho = (/\mu$ be the arrival rate, service rate and traffic intensity respectively.

II. System Equations

Assuming the limiting distribution of the number of customers in the system to be $\{P_n\}$. We obtain the following steady-state equations.

$$\lambda P_0 = \mu P_1 ,$$

$$((+\mu\ )P_{\downarrow}n = (P_{\downarrow}(n-1) + \mu P_{\downarrow}(n+1) , \qquad 1 \leq n < N ,$$

$$((+\mu\ )P_{\downarrow}N = (P_{\downarrow}(N-1) + 2\mu P_{\downarrow}(N+1) ,$$

$$((+2\mu\ )P_{\downarrow}n = (P_{\downarrow}(n-1) + 2\mu P_{\downarrow}(n+1) , \qquad N < n < 2N ,$$

$$((+2\mu\ )P_{\downarrow}2N = (P_{\downarrow}(2N-1) + 3\mu P_{\downarrow}(2N+1) ,$$

$$((+3\mu\ )P_{\downarrow}n = (P_{\downarrow}(n-1) + 3\mu P_{\downarrow}(n+1) , \qquad n > 2N .$$

Solution of the System Equations

This system of equations can be solved recursively, or by using the general solution for single Markovian queuing models which is given as follows

For $0<n\leq N$;

$$P_n = \prod_{i=0}^{n-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right) P_0$$

$$= (\lambda/\mu)^n P_0 \quad : \text{ (writing } \lambda/\mu = \rho \text{)}$$

$$= \rho^n P_0 \quad . \tag{2.1}$$

For $N<n\leq 2N$;

$$P_n = \prod_{i=0}^{n-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right) P_0$$

$$= \left[\prod_{i=0}^{N-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right) \prod_{i=N}^{n-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right) \right] P_0$$

$$= (\lambda/\mu)^N . (\lambda/2\mu)^{n-N} P_0$$

$$= \frac{1}{2^{n-N}} \rho^n P_0 \quad . \tag{2.2}$$

For $2N<n$;

$$P_n = \prod_{i=0}^{n-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right) P_0$$

$$= \left[\prod_{i=0}^{N-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right) \prod_{i=N}^{2N-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right) \prod_{i=2N}^{n-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right) \right] P_0$$

$$= (\lambda/\mu)^N . (\lambda/2\mu)^N . (\lambda/3\mu)^{n-2N} P_0$$

$$= \frac{1}{2^N 3^{n-2N}} \rho^n P_0 \quad . \tag{2.3}$$

Now, to determine P_0 we can use the normalizing condition $\sum_{n=0}^{\infty} P_n = 1$ and equations (2.1),(2.2) and (2.3) as follows

$$1 = \sum_{n=0}^{N} \rho^n P_0 + \sum_{n=N+1}^{2N} \frac{1}{2^{n-N}} \rho^n P_0 + \sum_{n=2N+1}^{\infty} \frac{1}{2^N 3^{n-2N}} \rho^n P_0$$

$$P_0^{-1} = \sum_{n=0}^{N} \rho^n + \sum_{n=N+1}^{2N} \frac{1}{2^{n-N}} \rho^n + \sum_{n=2N+1}^{\infty} \frac{1}{2^N 3^{n-2N}} \rho^n$$

$$= \sum_{n=0}^{N} \rho^n + \frac{\rho^{N+1}}{2} \sum_{n=0}^{N-1} \left(\frac{\rho}{2}\right)^n + \frac{\rho^{2N+1}}{3(2^N)} \sum_{n=0}^{\infty} \left(\frac{\rho}{3}\right)^n$$

$$= \frac{1-\rho^{N+1}}{1-\rho} + \frac{\rho^{N+1}\left(1-\left(\frac{\rho}{2}\right)^N\right)}{2-\rho} + \frac{\rho^{2N+1}}{2^N(3-\rho)}$$

$$= \frac{1}{1-\rho} - \frac{\rho^{N+1}}{(1-\rho)(2-\rho)} - \frac{\rho^{2N+1}}{2^N(2-\rho)(3-\rho)} \quad ,$$

$$P_0^{-1} = \frac{2-\rho-\rho^{N+1}}{(1-\rho)(2-\rho)} - \frac{\rho^{2N+1}}{2^N(2-\rho)(3-\rho)} \quad . \tag{2.4}$$

III. Profitable of using Additional Servers

Suppose the cost to the system due to the presence of a customer is $\$C_1$ and the cost of bringing in an additional server is $\$C_2$. It is profitable to use the additional server only if its cost is less than that due to the waiting customers see [4].

Let Q be the long-run queue length of the standard queue M/M/1 with the same parameters. It is evident that the M/M/1 queuing system with additional servers is recommended only if

$$C_1[E(Q)-E(Q_1)]>C_2[P_r(N<Q_1\leq 2N)+2P_r(2N<Q_1)], \tag{3.1}$$

where,

$E(Q)$=Expected number of customers in the system for the queuing model M/M/1 with no additional servers (standard).

$E(Q_1)$=Expected number of customers in the system for the queuing model M/M/1 with additional servers.

We have

$$E(Q) = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n \rho^n p_0 = \frac{\rho}{1-\rho}. \quad (3.2)$$

Also,

$$E(Q_1) = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{N} n \rho^n P_0 + \sum_{n=N+1}^{2N} n \frac{1}{2^{n-N}} \rho^n P_0 + \sum_{n=2N+1}^{\infty} n \frac{1}{2^N 3^{n-2N}} \rho^n P_0. \quad (3.3)$$

Now we have to find

$$\sum_{n=0}^{N} n\rho^n P_0 = P_0 \rho \sum_{n=0}^{N} n\rho^{n-1} = P_0 \rho \frac{d}{d\rho} \sum_{n=0}^{N} \rho^n = \frac{P_0 \rho}{(1-\rho)^2} [1+N\rho^{N+1} - (N+1)\rho^N]. \quad (3.4)$$

Similarly,

$$\sum_{n=N+1}^{2N} n \frac{1}{2^{n-N}} \rho^n P_0 = 2^N \rho P_0 \frac{d}{d\rho} \left[\left(\frac{\rho}{2}\right)^{N+1} \frac{1-\left(\frac{\rho}{2}\right)^N}{1-\frac{\rho}{2}} \right] = \frac{\rho^{N+1}}{(2-\rho)^2} P_0 \, [2(N+1) - 2^{-N+1}(2N+1)\rho^N - N\rho + 2^{-N+1} N\rho^{N+1}]. \quad (3.5)$$

And

$$\sum_{n=2N+1}^{\infty} n \frac{1}{2^N 3^{n-2N}} \rho^n P_0 = \frac{3^{2N+1}}{2^N} \rho P_0 \frac{d}{d\rho} \left[\left(\frac{\rho}{3}\right)^{2N+1} \frac{1}{3-\rho} \right] = \frac{\rho^{2N+1}}{2^N (3-\rho)^2} P_0 \, [3(2N+1) - 2N\rho]. \quad (3.6)$$

Substituting by (3.4) , (3.5) and (3.6) into (3.3) it follows that

$$E(Q_1) = \rho P_0 \Big\{ \frac{1}{(1-\rho)^2} [1+N\rho^{N+1} - (N+1)\rho^N] + \frac{\rho^N}{(2-\rho)^2} [2(N+1) - 2^{-N+1}(2N+1)\rho^N - N\rho + 2^{-N+1} N\rho^{N+1}] + \frac{\rho^{2N}}{2^N(3-\rho)^2} [3(2N+1) - 2N\rho] \Big\}$$
$$= \rho P_0 \left[A/(2^N (1-\rho)^2 (2-\rho)^2 (3-\rho)^2) \right], \quad (3.7)$$

where

$$A = 2^N (2-\rho)^2 (3-\rho)^2 [1+N\rho^{N+1} - (N+1)\rho^N] + 2^N (1-\rho)^2 (3-\rho)^2 \rho^N [2(N+1) - 2^{-N+1}(2N+1)\rho^N - N\rho + 2^{-N+1} N\rho^{N+1}] + (1-\rho)^2 (2-\rho)^2 \rho^{2N} [3(2N+1) - 2N\rho].$$

Also, we have to find the probability of the additional of one server,

$$P_r(N < Q_1 \le 2N) = \sum_{n=N+1}^{2N} P_n = \sum_{n=N+1}^{2N} \frac{\rho^n}{2^{n-N}} P_0$$

$$= P_0\, 2^N \sum_{n=N+1}^{2N} \left(\frac{\rho}{2}\right)^n$$

$$= P_0 \rho^{N+1} \frac{1-\left(\frac{\rho}{2}\right)^N}{(2-\rho)} \quad . \tag{3.8}$$

And the probability of the additional of two servers ,

$$P_r(2N < Q_1) = \sum_{n=2N+1}^{\infty} P_n$$

$$= \sum_{n=2N+1}^{\infty} \frac{1}{2^N 3^{n-2N}} \rho^n P_0$$

$$= \frac{3^{2N}}{2^N} P_0 \left(\frac{\rho}{3}\right)^{2N+1} \sum_{n=0}^{\infty} \left(\frac{\rho}{3}\right)^n$$

$$= P_0 \frac{\rho^{2N+1}}{2^N(3-\rho)} \quad . \tag{3.9}$$

Substituting by (3.2), (3.7), (3.8) and (3.9) into (3.1) it follows that the modified M/M/1 queuing system is recommended only when

$$\frac{C_2}{C_1} < \frac{E(Q)-E(Q_1)}{P_r(N < Q_1 \le 2N) + 2P_r(2N < Q_1)}$$

$$< \frac{\frac{\rho}{1-\rho} - \rho P_0 \left[\frac{A}{2^N(1-\rho)^2(2-\rho)^2(3-\rho)^2}\right]}{P_0\rho^{N+1}\frac{1-\left(\frac{\rho}{2}\right)^N}{(2-\rho)} + 2P_0\frac{\rho^{2N+1}}{2^N(3-\rho)}}$$

$$< \frac{\rho P_0\left[\frac{P_0^{-1}}{1-\rho} - \frac{A}{2^N(1-\rho)^2(2-\rho)^2(3-\rho)^2}\right]}{\rho P_0\left[\rho^N\frac{1-\left(\frac{\rho}{2}\right)^N}{(2-\rho)} + 2\frac{\rho^{2N}}{2^N(3-\rho)}\right]}$$

$$< \frac{\frac{P_0^{-1}}{1-\rho} - \frac{A}{2^N(1-\rho)^2(2-\rho)^2(3-\rho)^2}}{(3-\rho)\rho^N(2^N-\rho^N) + 2(2-\rho)\rho^{2N} \Big/ 2^N(2-\rho)(3-\rho)} \quad .$$

Substituting by (2.4), we get

$$\frac{C_2}{C_1} < \left[\frac{2-\rho-\rho^{N+1}}{(1-\rho)^2(2-\rho)} - \frac{\rho^{2N+1}}{2^N(1-\rho)(2-\rho)(3-\rho)} - \frac{A}{2^N(1-\rho)^2(2-\rho)^2(3-\rho)^2}\right]$$

$$\left[\frac{2^N(2-\rho)(3-\rho)}{(3-\rho)\rho^N(2^N-\rho^N) + 2(2-\rho)\rho^{2N}}\right]$$

$$< \left[\frac{2^N(3-\rho)(2-\rho-\rho^{N+1})}{(1-\rho)^2} - \frac{\rho^{2N+1}}{(1-\rho)} - \frac{A}{(1-\rho)^2(2-\rho)(3-\rho)}\right]$$

$$\left[\frac{1}{(3-\rho)\rho^N(2^N-\rho^N) + 2(2-\rho)\rho^{2N}}\right]$$

$$< \frac{1}{1-\rho}\left[\frac{2^N(2-\rho)(3-\rho)^2(2-\rho-\rho^{N+1}) - (1-\rho)(2-\rho)(3-\rho)\rho^{2N+1} - A}{(1-\rho)(2-\rho)(3-\rho)[(3-\rho)\rho^N(2^N-\rho^N) + 2(2-\rho)\rho^{2N}]}\right].$$

Substituting by A from equation (3.7)

$$\frac{C_2}{C_1} < \frac{1}{1-\rho}\Big\{ 2^N(2-\rho)(3-\rho)^2 \begin{bmatrix}(2-\rho-\rho^{N+1}) - (2-\rho) \\ (1+N\rho^{N+1}-(N+1)\rho^N)\end{bmatrix}$$

$$- (1-\rho)(2-\rho)\ [(3-\rho)\rho^{\dagger}(2N+1) + (1-\rho)(2-\rho)\rho^{\dagger}2N\ \ (3(2N+1) - 2N \vdash \rho)]$$

$$- 2^N(1-\rho)^2(3-\rho)^2\rho^N\ [2(N+1) - 2^{-N+1}(2N+1)\rho^N - N\rho + 2^{-N+1}N\rho^{N+1}]\Big\} \Big/$$

$$(1-\rho)(2-\rho)(3-\rho)[(3-\rho)\rho^N(2^N-\rho^N) + 2(2-\rho)\rho^{2N}] \ .$$

This can be simplified to

$$\frac{C_2}{C_1} < \frac{1}{(1-\rho)[(3-\rho)(2^N-\rho^N)+\ 2(2-\rho)\rho^N]} \Big\{ \frac{2^N(3-\rho)[2(N+1)-(3N+2)\rho+N\rho^2]}{(1-\rho)} - \frac{2^N(1-\rho)(3-\rho)[2(N+1)\text{-}\ 2^{-N+1}(2N+1)\rho^N - N\rho + 2^{-N+1}N\rho^{N+1}]}{(2-\rho)} - \frac{\rho^N[6(2N+1)-2(11N+3)\rho+(12N+5)\rho^2-2N\rho^3]}{(3-\rho)} \Big\}. \qquad (3.10)$$

From this inequality, depending on the decision variable, the largest value of C_2/C_1 can be determined when ρ and N are given. Table (1) has been obtained from the inequality (3.10) for the purpose of illustration. MATLAB program is used to perform these calculations.

Table (1)

/Nρ	0.1	0.2	0.3	0.4	0.5
1	4.1765	4.6119	5.1762	5.9293	6.9778
2	6.5671	7.4620	8.5872	10.0580	12.0807
3	8.8032	10.0205	11.5852	13.6639	16.5562
4	11.0264	12.5286	14.4718	17.0736	20.7233
5	13.2486	15.0296	17.3346	20.4265	24.7769

From this table, we can conclude that for any given value of N, the largest value of C_2/C_1 decreases with decreasing values of ρ, that is with increasing values of service rates. The basic advantage in such a system is the inherent flexibility of service. Then there exists an optimal stationary policy (resulting in the least long-run expected cost) characterized by a single positive finite number N such that it is optimal to use the slow service rate μ when the number of customers in the system is less than or equal to N, faster service rate 2μ when the number in the system is greater than N or less than or equal to 2N and more fast service rate 3μ when the number in the system is greater than 2N.

III. Maximum Likelihood Estimators of The Parameters

To find the likelihood function it is necessary to find the three basic components see [5].

1. the stationary distribution of number of units in the system with contribution $\lim_{t\to\infty} P_u(t) = P_u$,
2. the interarrival times of length t_i [each of which is exponential for n arrival units with contribution $\prod_{i=1}^{n}(e^{-(t_i}$)] ,
3. the service time of durations t_i for k units with contribution $\prod_{i=1}^{k}\mu e^{-\mu t_i}$) .

Hence, the likelihood function may be written as.

$$L(\theta) = \prod_{i=1}^{n}(e^{-(t_i}\prod_{i=1}^{k}\mu e^{-\mu t_i}P_u(u) \quad ; \quad \theta = [(,\mu] ,$$

$$= \prod_{i=1}^{n}(e^{-(t_i}\prod_{i=1}^{k}\mu e^{-\mu t_i}\frac{1}{2^N 3^{u-2N}}\rho^u P_0 \quad ;$$

(use equation (2.3))

$$= P_{\downarrow}0\ e^{\uparrow}(-"(" T)\ e^{\uparrow}(-\mu t)\ (^{\uparrow}(n+u)\ \mu^{\uparrow}(k-u)\ k_{\downarrow}1 ,$$

where ,

T=$\sum_{i=1}^{n} t_i$, t=$\sum_{i=1}^{k} t_i$ and k_1 is a constant .

Thus

$$ln[L(\theta)] = lnP_{\downarrow}0 - (T - \mu t + (n+u)ln(+(k-u)ln\mu + lnk_{\downarrow}1.$$

By partial differentiation with respect to (and μ , respectively, and equating the results by zeros, one can obtain

$$-(【[\partial lnL/\partial (]】_{\downarrow}(\theta=\theta\hat{}\,) = -T+\frac{n+u}{(\,)} = 0 \qquad (4.1)$$

and

$$\mu\left[\frac{\partial lnL}{\partial \mu}\right]_{\theta=\hat{\theta}} = \hat{\mu}\left[\frac{\partial lnP_0}{\partial \mu} - t + \frac{k-u}{\hat{\mu}}\right] = 0 \;, \qquad (4.2)$$

where $\hat{(}$ and QUOTE $\hat{\mu}\hat{\mu}$ are the maximum likelihood estimators for the parameters $\boldsymbol{\lambda}$ and μ , respectively.

To obtain $\hat{(}$ and $\hat{\mu}$ we need to use the following important theorem

THEOREM (4.1),

For the birth-death process with rates, namely $0 \leq (_{\downarrow}n = (g_{\downarrow}n$, $0 \leq \mu_n = \mu r_n$ for n positive integer and both g_n *and* r_n are bounded functions of n [1],

We have

$$E(n) = -(\,(\partial lnP_{\downarrow}0)/\partial (= \mu\frac{\partial lnP_0}{\partial \mu} \;,$$

where P_0 = delay probability and

E(n)= expected value of n .

Proof: It is clear that

$$P_n = P_{\downarrow}0\ \Pi_{\downarrow}(i=0)^{\uparrow}(n-1)\,(_{\downarrow}i/\mu_{\downarrow}(i+1) \qquad \forall n \geq 1$$

$$P_0 = 【[1+\Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,\prod_{\downarrow}(i=0)^{\uparrow}(n-1)\,(_{\downarrow}i/\mu_{\downarrow}(i+1)\,]】^{\uparrow}(-1)$$

Case (i): putting $(_{\downarrow}i = (g_{\downarrow}i$, it follows that

$$P_0 = 【[1+\Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,【(^{\uparrow}n\ \Pi_{\downarrow}(i=0)^{\uparrow}(n-1)\,g_{\downarrow}i/\mu_{\downarrow}(i+1)\,】]】^{\uparrow}(-1) \;.$$

Taking the logarithm of both sides of this equation and differentiating the resulting equation partially with respect to (*yield*

$$-(\partial lnP_{\downarrow}0)/\partial (= P_{\downarrow}0\ [\Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,【n(^{\uparrow}(n-1)\ \Pi_{\downarrow}(i=0)^{\uparrow}(n-1)\,g_{\downarrow}i/\mu_{\downarrow}(i+1)\,】] \;.$$

But,

$$E(n) = \Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,【nP_{\downarrow}n = P_{\downarrow}0\ [\Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,【n\prod_{\downarrow}(i=0)^{\uparrow}(n-1)\,(_{\downarrow}i/\mu_{\downarrow}(i+1)\,】]】 \;.$$

Therefore, it follows that

$$E(n) = -(\,(\partial lnP_{\downarrow}0)/\partial (\;.$$

Case (ii): putting $\mu_i = \mu r_i$, it follows that

$$P_0 = 【[1+\Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,【1/\mu^{\uparrow}n\ \Pi_{\downarrow}(i=0)^{\uparrow}(n-1)\,(_{\downarrow}i/r_{\downarrow}(i+1)\,】]】^{\uparrow}(-1) \;.$$

Taking the logarithm of both sides of this equation, it follows that

$$lnP_0 = -ln[1+\Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,【1/\mu^{\uparrow}n\ \Pi_{\downarrow}(i=0)^{\uparrow}(n-1)\,(_{\downarrow}i/r_{\downarrow}(i+1)\,】] \;.$$

Therefore,

$$\frac{\partial lnP_0}{\partial \mu} = P_{\downarrow}0\ [\Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,【n/\mu^{\uparrow}(n+1)\ \Pi_{\downarrow}(i=0)^{\uparrow}(n-1)\,(_{\downarrow}i/r_{\downarrow}(i+1)\,】] \;.$$

Hence,

$$\mu\frac{\partial lnP_0}{\partial \mu} = P_{\downarrow}0\ [\Sigma_{\downarrow}(n=1)^{\uparrow}\infty\,【n/\mu^{\uparrow}n\ \prod_{\downarrow}(i=0)^{\uparrow}(n-1)\,(_{\downarrow}i/r_{\downarrow}(i+1)\,】] \;.$$

Thus, it is evident that

$$E(n) = \mu\frac{\partial lnP_0}{\partial \mu} \;,$$

Hence, the theorem is proved.

Using theorem (4.1), equation (4.1) can be written in the form

$$-(\hat{}\,(\partial lnP_{\downarrow}0)/\partial (+(\hat{}T-(n+u)=0 \;,$$

$$E\hat{}(n) + (\hat{}T-(n+u)=0 \;,$$

$$\hat{(} = \frac{1}{T\left[(n+u)-\hat{E}(n)\right]} \;. \qquad (4.3)$$

Also, by using theorem (4.1), equation (4.2) can be written in the form

$$\frac{\hat{\mu}(\partial lnP_0)}{\partial \mu} - \hat{\mu}t + (k-u) = 0 \;,$$

$$\hat{E}(n) - \hat{\mu}t + (k-u) = 0 \;,$$

$$\hat{\mu} = \frac{1}{t[(k-u)+\hat{E}(n)]} \quad . \tag{4.4}$$

Where $\hat{E}(n)$ is the maximum likelihood estimator of E(n) .

Obtaining the value of $\hat{E}(n)$ from each of the equations in (4.3), (4.4) and dividing the resulting values yields

From (4.3)

$$T(\hat{} = (n+u) - E\hat{}(n),$$

$$E\hat{}(n) = (n+u) - T(\hat{} \;.$$

From (4.4)

$$t\hat{\mu} = (k-u) + \hat{E}(n) \;,$$

$$\hat{E}(n) = t\hat{\mu} - (k-u) \;,$$

$$(t\mu\hat{} - (k-u))/(-T(\hat{} + (n+u)) = 1 \;. \tag{4.5}$$

Recalling that $\hat{\rho}$, the maximum likelihood estimator of the traffic intensity ρ , is given by $\rho\hat{} = (\hat{}/\mu\hat{}$ and using equations (4.3), (4.4)

$$\rho\hat{} = (\hat{}/\mu\hat{} \;=\; (t[(n+u) - E\hat{}(n)])/(T[(k-u) + E\hat{}(n)]) \tag{4.6}$$

This equation can be written in the form

$$t[n+u-\hat{E}(n)] - \hat{\rho}T[k-u+\hat{E}(n)] = 0 \;. \tag{4.7}$$

Consider the special case M/M/1 on replacing $\hat{E}(n)$ by $\frac{\hat{\rho}}{1-\hat{\rho}}$ into equation (4.7)

$$t\left[n+u-\frac{\hat{\rho}}{1-\hat{\rho}}\right] - \hat{\rho}T\left[k-u+\frac{\hat{\rho}}{1-\hat{\rho}}\right] = 0 \;,$$

$$t[(n+u)(1-\hat{\rho}) - \hat{\rho}] - \hat{\rho}T[(k-u)(1-\hat{\rho}) + \hat{\rho}] = 0 \;,$$

$$T(k-u-1)\hat{\rho}^2 - [t(n+u+1) + T(k-u)]\hat{\rho} + t(n+u) = 0 \;. \tag{4.8}$$

Replacing (k-u-1) and (n+u+1) by (k-u) and (n+u), respectively, in (4.8) yields.

$$T(k-u)\hat{\rho}^2 - [t(n+u) + T(k-u)]\hat{\rho} + t(n+u) = 0 \;,$$

$$T(k-u)\hat{\rho}(\hat{\rho}-1) - t(n+u)(\hat{\rho}-1) = 0 \;,$$

$$(\hat{\rho}-1)[T(k-u)\hat{\rho} - t(n+u)] = 0 \;, \tag{4.9}$$

$$T(k-u)\hat{\rho} - t(n+u) = 0 \;.$$

Then

$$\hat{\rho} = \frac{t(n+u)}{T(k-u)} \;, \tag{4.10}$$

is a solution for the modified equation (4.9) of degree two in $\hat{\rho}$. Using (4.5) and (4.10) it follows that

$$\rho\hat{} = t(n+u)/T(k-u) \;*\; (t\mu\hat{} - (k-u))/(-T(\hat{} + (n+u)) \;,$$

$$\rho\hat{} = (t{\uparrow}2\,\mu\hat{}(n+u) - t(k-u)(n+u))/(-T{\uparrow}2\,(\hat{}(k-u) + T(k-u)(n+u)) \;,$$

$$-T{\uparrow}2\,\rho\hat{}(\hat{}(k-u) + T\rho\hat{}(k-u)(n+u) = t{\uparrow}2\,\mu\hat{}(n+u) - t(k-u)(n+u) \;,$$

$$t{\uparrow}2\,\mu\hat{}(n+u) + T{\uparrow}2\,\rho\hat{}(\hat{}(k-u) = (n+u)(k-u)[T\rho\hat{} + t]. \tag{4.11}$$

To find $\hat{\mu}$, put $(\hat{} = \rho\hat{}\mu\hat{}$ into equation (4.11)

$$t^2\hat{\mu}(n+u) + T^2\hat{\rho}^2\hat{\mu}(k-u) = (n+u)(k-u)[T\hat{\rho} + t],$$

$$\hat{\mu} = \frac{(n+u)(k-u)[T\hat{\rho}+t]}{t^2(n+u) + T^2\hat{\rho}^2(k-u)} \qquad \text{(use equation(4.10))}$$

$$= \frac{(\mathbf{n}+\mathbf{u})(\mathbf{k}-\mathbf{u})\left[\mathbf{T}\frac{\mathbf{t}(\mathbf{n}+\mathbf{u})}{\mathbf{T}(\mathbf{k}-\mathbf{u})} + \mathbf{t}\right]}{\mathbf{t}^2(\mathbf{n}+\mathbf{u}) + \mathbf{T}^2\frac{\mathbf{t}^2(\mathbf{n}+\mathbf{u})^2}{\mathbf{T}^2(\mathbf{k}-\mathbf{u})^2}(\mathbf{k}-\mathbf{u})}$$

$$= \frac{\mathbf{t}(\mathbf{n}+\mathbf{u})[\mathbf{n}+\mathbf{u}+\mathbf{k}-\mathbf{u}]}{\mathbf{t}(\mathbf{n}+\mathbf{u})\left[\mathbf{t}+\frac{\mathbf{t}(\mathbf{n}+\mathbf{u})}{\mathbf{k}-\mathbf{u}}\right]}$$

$$= \frac{\mathbf{n}+\mathbf{k}}{\mathbf{t}+\mathbf{T}\frac{\mathbf{t}(\mathbf{n}+\mathbf{u})}{\mathbf{T}(\mathbf{k}-\mathbf{u})}} \;. \qquad \text{(from equation(4.10))}$$

Thus

$$\hat{\mu} = \frac{n + k}{t + T\hat{\rho}} \quad . \tag{4.12}$$

To find $\hat{\lambda}$ put $\hat{\mu} = \hat{\lambda}/\hat{\rho}$ into equation (4.12)

$$\hat{\lambda}/\hat{\rho} = (n + k)/(T\hat{\rho} + t) \quad ,$$

$$\hat{\lambda} = \frac{\hat{\rho}(n + k)}{T\hat{\rho} + t} \quad . \tag{4.13}$$

Then, by using (4.10) it follows that

$$\hat{\mu} = \frac{n + k}{T\hat{\rho} + t} = \frac{k - u}{t} \quad ,$$

$$\hat{\lambda} = (\hat{\rho}(n + k))/(T\hat{\rho} + t) = (n + u)/T \quad . \tag{4.14}$$

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