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A Note on a Three Variables Analogue of Bessel Polynomials

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Abstract: *The present paper deals with a study of a three variables analogue of Bessel polynomials. Certain representations, a Schlafli's contour integral, a fractional integral, Laplace transformations, some generating functions and double and triple generating functions have been obtained.*

I. Introduction

In 1949 Krall and Frink [12] initiated a study of simple Bessel polynomial

$$Y_n(x) = {}_2F_0\left[-n, 1+n; -; -\frac{x}{2}\right] \qquad (1.1)$$

and generalized Bessel polynomial

$$Y_n(a, b, x) = {}_2F_0\left[-n, a-1+n; -; -\frac{x}{b}\right] \qquad (1.2)$$

These polynomials were introduced by them in connection with the solution of the wave equation in spherical coordinates. They are the polynomial solutions of the differential equation.

$$x^2 y''(x) + (ax+b)\, y'(x) = n\,(n+a-1)\, y(x) \qquad (1.3)$$

where n is a positive integer and a and b are arbitrary parameters. These polynomials are orthogonal on the unit circle with respect to the weight function

$$\rho(x, \alpha) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+n-1)} \left(-\frac{2}{x}\right)^n . \qquad (1.4)$$

Several authors including Agarwal [1], Al-Salam [2], Brafman [3], Burchnall [4], Carlitz [5], Chatterjea [6], Dickinson [7], Eweida [9], Grosswald [10], Rainville [15] and Toscano [19] have contributed to the study of the Bessel polynomials.

Recently in the year 2000, Khan and Ahmad [11] studied two variables analogue $Y_n^{(\alpha,\beta)}(x, y)$ of the Bessel polynomials $Y_n^{(\alpha)}(x)$ defined by

$$Y_n^{(\alpha)}(x) = {}_2F_0\left[-n, \alpha+n+1; -; -\frac{x}{2}\right] \qquad (1.5)$$

The aim of the present paper is to introduce a three variables analogue $Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c)$ of (1.2) and to obtain certain results involving the three variables Bessel polynomial $Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c)$.

II. The Polynomials

$Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c)$**:** The Bessel polynomial of three variables $Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c)$ is defined as follows:

$$Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c)$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{j=0}^{n-r-s} \frac{(-n)_{r+s+j} (\alpha+n+1)_j (\beta+n+1)_s (\gamma+n+1)_r}{r!\, s!\, j!} \left(-\frac{x}{a}\right)^j \left(-\frac{y}{b}\right)^s \left(-\frac{z}{c}\right)^r \qquad (2.1)$$

For z = 0, a = b = 2, (2.1) reduces to the two variables analogue $Y_n^{(\alpha,\beta)}(x, y)$ of Bessel polynomials (1.5) as given below :

$$Y_n^{(\alpha,\beta,\gamma)}(x, y, 0; 2, 2, c) = Y_n^{(\alpha,\beta)}(x, y) \qquad (2.2)$$

Similarly

$$Y_n^{(\alpha,\beta,\gamma)}(x, 0, z; 2, b, 2) = Y_n^{(\alpha,\gamma)}(x, z) \qquad (2.3)$$

$$Y_n^{(\alpha,\beta,\gamma)}(0, y, z; a, 2, 2) = Y_n^{(\beta,\gamma)}(y, z) \tag{2.4}$$

Also for $\alpha = -n-1$, and $b = c = 2$

$$Y_n^{(-n-1,\beta,\gamma)}(x, y, z; a, 2, 2) = Y_n^{(\beta,\gamma)}(y, z) \tag{2.5}$$

Similarly,

$$Y_n^{(\alpha,-n-1,\gamma)}(x, y, z; 2, b, 2) = Y_n^{(\alpha,\gamma)}(x, z) \tag{2.6}$$

$$Y_n^{(\alpha,\beta,-n-1)}(x, y, z; 2, 2, c) = Y_n^{(\alpha,\beta)}(x, y) \tag{2.7}$$

where

$$Y_n^{(\alpha,\beta)}(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\alpha+n+1)_s(\beta+n+1)_r}{r!\, s!} \left(-\frac{x}{2}\right)^s \left(-\frac{y}{2}\right)^r \tag{2.8}$$

Also, for $y = z = 0$, $a = 2$, (2.1) reduces to the Bessel polynomials $Y_n^{(\alpha)}(x)$ as given below:

$$Y_n^{(\alpha,\beta,\gamma)}(x, 0, 0; 2, b, c) = Y_n^{(\alpha)}(x) \tag{2.9}$$

where $Y_n^{(\alpha)}(x)$ is defined by (1.5).

Similarly

$$Y_n^{(\alpha,\beta,\gamma)}(0, y, 0; a, 2, c) = Y_n^{(\beta)}(y) \tag{2.10}$$

$$Y_n^{(\alpha,\beta,\gamma)}(0, 0, z; 2, 2, c) = Y_n^{(\gamma)}(z) \tag{2.11}$$

Also, for $\beta = \gamma = -n-1$, $a = 2$, we have

$$Y_n^{(\alpha,-n-1,-n-1)}(x, y, z; 2, b, c) = Y_n^{(\alpha)}(x) \tag{2.12}$$

Similarly

$$Y_n^{(-n-1,\beta,-n-1)}(x, y, z; a, 2, c) = Y_n^{(\beta)}(y) \tag{2.13}$$

$$Y_n^{(-n-1,-n-1,\gamma)}(x, y, z; a, b, 2) = Y_n^{(\gamma)}(z) \tag{2.14}$$

III. Integral Representations

It is easy to show that the polynomial $Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c)$ has the following integral representations:

$$\frac{1}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\gamma+n+1)} \int_0^\infty \int_0^\infty \int_0^\infty u^{\alpha+n} v^{\beta+n} w^{\gamma+n} \left(1 + \frac{xu}{a} + \frac{yv}{b} + \frac{zw}{c}\right)^n e^{-u-v-w}\, du\, dv\, dw$$
$$= Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c) \tag{3.1}$$

For $z = 0$, $a = b = 2$, (3.1) reduces to

$$\frac{1}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} \int_0^\infty \int_0^\infty u^{\alpha+n} v^{\beta+n} \left(1 + \frac{xu}{2} + \frac{yv}{2}\right)^n e^{-u-v}\, du\, dv$$
$$= Y_n^{(\alpha,\beta)}(x, y) \tag{3.2}$$

a result due to Khan and Ahmad [11].

For $y = z = 0$, α replaced by $a-2$ and a replaced by b, (3.1) becomes

$$Y_n(a, b, x) = \frac{1}{\Gamma(a-1+n)} \int_0^\infty t^{a-2+n} \left(1 + \frac{xt}{b}\right)^n e^{-t}\, dt \tag{3.3}$$

a result due to Agarwal [1].

$$\int_0^t \int_0^s \int_0^r x^\alpha (r-x)^{n-1} y^\beta (s-y)^{n-1} z^\gamma (t-z)^{n-1} Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c)\, dx\, dy\, dz$$
$$= \frac{r^{\alpha+n} s^{\beta+n} t^{\gamma+n} \{\Gamma(n)\}^3}{(\alpha+1)_n (\beta+1)_n (\gamma+1)_n} Y_n^{(\alpha-n,\beta-n,\gamma-n)}(r, s, t; a, b, c) \tag{3.4}$$

$$\int_0^1 \int_0^1 \int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(xu, yvzw; a, b, c)\, du\, dv\, dw$$

$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: -; -; - : \alpha+n+1, \delta; \beta+n+1, \eta; \gamma+n+1, \xi; \\ - :: -; -; - : \quad \delta+\lambda \quad ; \quad \eta+\mu \quad ; \quad \xi+\nu \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.5}$$

where $F^{(3)}$ [ ] is in the form of a general triple hypergeometric series $F^{(3)}$ [x, y, z] (cf. Srivastava [18], p. 428).

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(x(1-u), yvzw; a, b, c)\,du\,dv\,dw$$
$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: -; -; - : \alpha+n+1, \lambda; \beta+n+1, \eta; \gamma+n+1, \xi; \\ - :: -; -; - : \quad \delta+\lambda \quad ; \quad \eta+\mu \quad ; \quad \xi+\nu \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.6}$$

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(xu, y(1-v), zw; a, b, c)\,du\,dv\,dw$$
$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: -; -; - : \alpha+n+1, \delta; \beta+n+1, \mu; \gamma+n+1, \xi; \\ - :: -; -; - : \quad \delta+\lambda \quad ; \quad \eta+\mu \quad ; \quad \xi+\nu \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.7}$$

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(xu, yv, z(1-w); a, b, c)\,du\,dv\,dw$$
$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: -; -; - : \alpha+n+1, \delta; \beta+n+1, \eta; \gamma+n+1, \nu; \\ - :: -; -; - : \quad \delta+\lambda \quad ; \quad \eta+\mu \quad ; \quad \xi+\nu \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.8}$$

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(x(1-u), y(1-v), zw; a, b, c)\,du\,dv\,dw$$
$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: -; -; - : \alpha+n+1, \lambda; \beta+n+1, \mu; \gamma+n+1, \xi; \\ - :: -; -; - : \quad \delta+\lambda \quad ; \quad \eta+\mu \quad ; \quad \xi+\nu \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.9}$$

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(x(1-u), yv, z(1-w); a, b, c)\,du\,dv\,dw$$
$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: -; -; - : \alpha+n+1, \lambda; \beta+n+1, \eta; \gamma+n+1, \nu; \\ - :: -; -; - : \quad \delta+\lambda \quad ; \quad \eta+\mu \quad ; \quad \xi+\nu \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.10}$$

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(xu, y(1-v), z(1-w); a, b, c)\,du\,dv\,dw$$
$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: -; -; - : \alpha+n+1, \delta; \beta+n+1, \mu; \gamma+n+1, \nu; \\ - :: -; -; - : \quad \delta+\lambda \quad ; \quad \eta+\mu \quad ; \quad \xi+\nu \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.11}$$

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(x(1-u), y(1-v), z(1-w); a, b, c)\,du\,dv\,dw$$
$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: -; -; - : \alpha+n+1, \lambda; \beta+n+1, \mu; \gamma+n+1, \nu; \\ - :: -; -; - : \quad \delta+\lambda \quad ; \quad \eta+\mu \quad ; \quad \xi+\nu \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.12}$$

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1} v^{\eta-1}(1-v)^{\mu-1} w^{\xi-1}(1-w)^{\nu-1} Y_n^{(\alpha,\beta,\gamma)}(xvw, yuw, zuv; a, b, c)\,du\,dv\,dw$$
$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)} F^{(3)}\left[\begin{array}{l} -n :: \delta \;;\; \eta \;;\; \xi : \alpha+n+1; \beta+n+1; \gamma+n+1; \\ - :: \delta+\lambda \;; \eta+\mu \;; \xi+\nu : \quad -; \quad -; \quad - \quad ; \end{array} -\frac{x}{a}, -\frac{y}{b}, -\frac{z}{c}\right] \tag{3.13}$$

$$\int_0^1\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1}v^{\eta-1}(1-v)^{\mu-1}w^{\xi-1}(1-w)^{\nu-1}Y_n^{(\alpha,\beta,\gamma)}(xv(1-w),yw(1-u),zu(1-v);a,b,c)\,du\,dv\,dw$$

$$=\frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)\Gamma(\xi)\Gamma(\nu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)\Gamma(\xi+\nu)}F^{(3)}\left[\begin{matrix}-n::-;-;-:\alpha+n+1,\delta,\mu;\beta+n+1,\lambda,\xi;\gamma+n+1,\eta,\nu; \\ -::\ \delta+\lambda\ ;\ \eta+\mu\ ;\ \xi+\nu\ :\ -;\ -;\ -\ ;\end{matrix}-\frac{x}{a},-\frac{y}{b},-\frac{z}{c}\right] \tag{3.14}$$

Particular Cases:

Some interesting particular cases of the above results are as follows :

(i) Taking $\delta = \alpha + 1$, $\eta = \beta + 1$, $\xi = \gamma + 1$, $\lambda = \mu = \nu = n$ in (3.5), we obtain

$$\int_0^1\int_0^1\int_0^1 u^{\alpha}(1-u)^{n-1}v^{\beta}(1-v)^{n-1}w^{\gamma}(1-w)^{n-1}Y_n^{(\alpha,\beta,\gamma)}(xu,yvzw;a,b,c)\,du\,dv\,dw$$

$$=\frac{\{\Gamma(n)\}^3}{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}Y_n^{(\alpha-n,\beta-n,\gamma-n)}(x,y,z;a,b,c) \tag{3.15}$$

which is equivalent to (3.4)

(ii) Taking $\delta = \eta = \xi = n + 1$, $\lambda = \alpha$, $\mu = \beta$, $\nu = \gamma$ in (3.5), we get

$$\int_0^1\int_0^1\int_0^1 u^{n}(1-u)^{\alpha-1}v^{n}(1-v)^{\beta-1}w^{n}(1-w)^{\gamma-1}Y_n^{(\alpha,\beta,\gamma)}(xu,yvzw)\,du\,dv\,dw$$

$$=\frac{(n!)^3}{(\alpha)_{n+1}(\beta)_{n+1}(\gamma)_{n+1}}Y_n^{(0,0,0)}(x,y,z) \tag{3.16}$$

(iii) Replacing δ by $\alpha + n + 1 - \delta$, η by $\beta + n + 1 - \eta$, ξ by $\gamma + n + 1 - \xi$, and putting $\lambda = \delta$, $\mu = \eta$ and $\nu = \xi$ in (3.5), we get

$$\int_0^1\int_0^1\int_0^1 u^{\alpha-\delta+n}(1-u)^{\delta-1}v^{\beta-\eta+n}(1-v)^{\eta-1}w^{\gamma-\xi+n}(1-w)^{\xi-1}Y_n^{(\alpha,\beta,\gamma)}(xu,yvzw;a,b,c)\,du\,dv\,dw$$

$$=\frac{\Gamma(\alpha-\delta+n+1)\Gamma(\delta)\Gamma(\beta-\eta+n+1)\Gamma(\eta)\Gamma(\gamma-\xi+n+1)\Gamma(\xi)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}Y_n^{(\alpha-\delta,\beta-\eta,\gamma-\xi)}(x,y,z;a,b,c) \tag{3.17}$$

Similar particular cases hold for (3.6), (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12).

(iv) For $z = 0$, $a = b = 2$, results (3.4), (3.5), (3.6), (3.7) and (3.9) become

$$\int_0^s\int_0^r x^{\alpha}(r-x)^{n-1}y^{\beta}(s-y)^{n-1}Y_n^{(\alpha,\beta)}(x,y)\,dx\,dy$$

$$=\frac{r^{\alpha+n}s^{\beta+n}t^{\gamma+n}\{\Gamma(n)\}^2}{(\alpha+1)_n(\beta+1)_n}Y_n^{(\alpha-n,\beta-n)}(r,s) \tag{3.18}$$

$$\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1}v^{\eta-1}(1-v)^{\mu-1}Y_n^{(\alpha,\beta)}(xu,yv)\,du\,dv$$

$$=\frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)}F^{1:2;2}_{-:1;1}\left[\begin{matrix}-n:\ \alpha+n+1,\delta;\ \beta+n+1,\eta; \\ -:\ \delta+\lambda\ ;\ \eta+\mu\ ;\end{matrix}-\frac{x}{2},-\frac{y}{2}\right] \tag{3.19}$$

$$\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1}v^{\eta-1}(1-v)^{\mu-1}Y_n^{(\alpha,\beta)}(x(1-u),yv)\,du\,dv$$

$$=\frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)}F^{1:2;2}_{-:1;1}\left[\begin{matrix}-n:\ \alpha+n+1,\lambda;\ \beta+n+1,\eta; \\ -:\ \delta+\lambda\ ;\ \eta+\mu\ ;\end{matrix}-\frac{x}{2},-\frac{y}{2}\right] \tag{3.20}$$

$$\int_0^1\int_0^1 u^{\delta-1}(1-u)^{\lambda-1}v^{\eta-1}(1-v)^{\mu-1}Y_n^{(\alpha,\beta)}(x(1-u),y(1-v))\,du\,dv$$

$$=\frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\eta)\Gamma(\mu)}{\Gamma(\delta+\lambda)\Gamma(\eta+\mu)}F^{1:2;2}_{-:1;1}\left[\begin{matrix}-n:\ \alpha+n+1,\lambda;\ \beta+n+1,\mu; \\ -:\ \delta+\lambda\ ;\ \eta+\mu\ ;\end{matrix}-\frac{x}{2},-\frac{y}{2}\right] \tag{3.21}$$

Results (3.18), (3.19), (3.20) and (3.21) are due to Khan and Ahmad [11].
Also, using the integral (see Erdelyi et al. [8], vol. I, p. 14),

$$2\,i \sin \pi z\, \Gamma(z) = -\int_{\infty}^{(0+)} (-t)^{z-1}\, e^{-t}\, dt \tag{3.22}$$

and the fact that

$$(1-x-y-z)^n = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{j=0}^{n-r-s} \frac{(-n)_{r+s+j}\, x^j\, y^s\, z^r}{r!\;\; s!\;\; j!} \tag{3.23}$$

we can easily derive the following integral representations for $Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c)$:

$$-\int_{\infty}^{(0+)} \int_{\infty}^{(0+)} \int_{\infty}^{(0+)} (-u)^{\alpha+n} (-v)^{\beta+n} (-w)^{\gamma+n}\, e^{-u-v-w} \left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n du\, dv\, dw$$
$$= 8\,i\,(-1)^n \sin \pi\alpha \sin \pi\beta \sin \pi\gamma\, \Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\gamma+n+1)\, Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c) \tag{3.24}$$

$$\frac{(-1)^{n+1} \sin \pi\alpha \sin \pi\beta \sin \pi\gamma\, \Gamma(1+\alpha+n)\Gamma(1+\beta+n)\Gamma(1+\gamma+n)}{\pi^3}$$
$$\int_0^{\infty} \int_0^{\infty} \int_0^{\infty} u^{-\alpha-n-1}\, v^{-\beta-n-1}\, w^{-\gamma-n-1}\, e^{-u-v-w}\, Y_n^{(\alpha,\beta,\gamma)}\left(\frac{ax}{u}, \frac{by}{v}, \frac{cz}{w}; a, b, c\right) du\, dv\, dw$$
$$= (1-x-y-z)^n \tag{3.25}$$

IV. Schlafli's Contour Integral

It is easy to show that

$$\int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} u^{\alpha+n}\, v^{\beta+n}\, w^{\gamma+n}\, e^{u+v+w} \left(1-\frac{xu}{a}-\frac{yv}{b}-\frac{zw}{c}\right)^n du\, dv\, dw$$
$$= 8\,i\,(-1)^n \sin \pi\alpha \sin \pi\beta \sin \pi\gamma\, \Gamma(1+\alpha+n)\Gamma(1+\beta+n)\Gamma(1+\gamma+n)\, Y_n^{(\alpha,\beta,\gamma)}(x, y, z; a, b, c) \tag{4.1}$$

Proof of (4.1) : We have

$$\frac{1}{(2\pi i)^3} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} u^{\alpha+n}\, v^{\beta+n}\, w^{\gamma+n}\, e^{u+v+w} \left(1-\frac{xu}{a}-\frac{yv}{b}-\frac{zw}{c}\right)^n du\, dv\, dw$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{j=0}^{n-r-s} \frac{(-n)_{r+s+j}}{r!\,s!\,j!} \left(\frac{x}{a}\right)^j \left(\frac{y}{b}\right)^s \left(\frac{z}{c}\right)^r \frac{1}{(2\pi i)^3} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} u^{\alpha+n+j} v^{\beta+n+s} w^{\gamma+n+r} e^{u+v+w} du\, dv\, dw$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{j=0}^{n-r-s} \frac{(-n)_{r+s+j} \left(\frac{x}{a}\right)^j \left(\frac{y}{b}\right)^s \left(\frac{z}{c}\right)^r}{r!\,s!\,j!\; \Gamma(-\alpha-n-j)\Gamma(-\beta-n-s)\;\Gamma(-\gamma-n-r)}$$

using Hankel's formula (see A. Eerdelyi et al. [8], 1.6 (2)).

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t\, t^{-z}\, dt \tag{4.2}$$

Finally (4.1) follows from (2.1) after using the result

$$\Gamma(z)\Gamma(1-z) \;=\; \pi \;\operatorname{cosec} \pi z \tag{4.3}$$

for z = 0, a = b = 2, (4.1) reduces to

$$\int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} u^{\alpha+n}\, v^{\beta+n}\, e^{u+v+w} \left(1-\frac{xu}{a}-\frac{yv}{b}\right)^n du\, dv$$
$$= -4 \sin \pi\alpha \sin \pi\beta\, \Gamma(1+\alpha+n)\Gamma(1+\beta+n)\, Y_n^{(\alpha,\beta)}(x, y) \tag{4.4}$$

which is due to Khan and Ahmad [11].

V. Fractional Integrals

Let L denote the linear space of (equivalent classes of) complex – valued functions f(x) which are Lebesgue – integrable on [0, α], α < ∞. For f(x) εL and complex number μ with Rl μ > 0, the Riemann – Liouville fractional integral of order μ is defined as (see Prabhakar [13], p. 72)

$$I^{\mu}\ f(x)=\frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}\ f(t)\ dt \qquad \text{for almost all } x\ \varepsilon\ [0,\alpha] \tag{5.1}$$

Using the operator I^{μ}, Prabhakar [14] obtained the following result for Rl μ > 0 and Rl α > –1.

$$I^{\mu}\left[x^{\alpha}\ Z_n^{\alpha}\ (x;k)\right]=\frac{\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha+\mu+1)}\ x^{\alpha+\mu}\ Z_n^{\alpha+\mu}\ (x;k) \tag{5.2}$$

where $Z_n^{\alpha}\ (x;k)$ is Konhauser's biorthozonal polynomial.

Khan and Ahmad [11] defined a two variable analogue of (5.1) by means of the following relation :

$$I^{\lambda,\mu}\left[f\ (x,y)\right]=\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_0^x\int_0^y (x-u)^{\lambda-1}\ (y-v)^{\mu-1}\ f(u,v)\,du\,dv \tag{5.3}$$

and obtained the following result :

$$I^{\lambda,\mu}\left[x^{\alpha+n-\lambda}\ y^{\beta+n-\mu}\ Y_n^{(\alpha,\beta)}(x,y)\right]=\frac{x^{\alpha+n}\ y^{\beta+n}\ \Gamma(\alpha-\lambda+n+1)\Gamma(\beta-\mu+n+1)}{\Gamma(\alpha+n+1)\ \Gamma(\beta+n+1)}\ Y_n^{(\alpha-\lambda,\beta-\mu)}\ (x,y) \tag{5.4}$$

In an attempt to obtained a result analogous to (5.4) for the polynomial $Y_n^{(\alpha,\beta,\gamma)}\ (x,y,z;a,b,c)$ we first seek a three variable analogue of (5.1).

A three variable analogue of I^{μ} may be defined as

$$I^{\lambda,\mu,\eta}\left[f\ (x,y,z)\right]=\frac{1}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\eta)}\int_0^x\int_0^y\int_0^z (x-u)^{\lambda-1}\ (y-v)^{\mu-1}(z-w)^{\eta-1}\ f(u,v,w)\,du\,dv\,dw \tag{5.5}$$

Putting $f(x,y,z)=x^{\alpha+n-\lambda}\ y^{\beta+n-\mu}\ z^{\gamma+n-\eta}\ Y_n^{(\alpha,\beta,\gamma)}\ (x,y,z;a,b,c)$ in (5.5), we obtain

$$I^{\lambda,\mu,\eta}=\left[x^{\alpha+n-\lambda}\ y^{\beta+n-\mu}\ z^{\gamma+n-\eta}\ Y_n^{(\alpha,\beta,\gamma)}\ (x,y,z;a,b,c)\right]$$

$$=\frac{x^{\alpha+n}\ y^{\beta+n}\ z^{\gamma+n}\ \Gamma(\alpha-\lambda+n+1)\Gamma(\beta-\mu+n+1)\Gamma(\gamma-\eta+n+1)}{\Gamma(\alpha+n+1)\ \Gamma(\beta+n+1)\Gamma(\gamma+n+1)}\ Y_n^{(\alpha-\lambda,\beta-\mu,\gamma-\eta)}\ (x,y,z;a,b,c) \tag{5.6}$$

VI. Laplace Transform

In the usual notation the Laplace transform is given by

$$L\{f(t):s\}=\int_0^{\infty} e^{-st}\ f(t)\,dt,\ Rl\,(s-a)>0 \tag{6.1}$$

where f ε L (0, R) for every R > 0 and f(t) = 0(e^{at}), t → ∞.

Khan and Ahmad [11] introduced a two variable analogue of (6.1) by means of the relation:

$$L\{f(u,v):s_1,s_2\}=\int_0^{\infty}\int_0^{\infty} e^{-s_1u-s_2v}\ f(u,v)\,du\,dv \tag{6.2}$$

and established the following results :

$$L\left\{u^{-\alpha-n-1}\ v^{-\beta-n-1}\ Y_n^{(\alpha,\beta)}\left(\frac{2x}{us_1},\frac{2y}{vs_2}\right):s_1,s_2\right\}$$

$$=\frac{\pi^2\ s_1^{\alpha+n}\ s_2^{\beta+n}}{\sin\pi\alpha\ \sin\pi\beta\ \Gamma(\alpha+n+1)\Gamma(\beta+n+1)}(1-x-y)^n \tag{6.3}$$

and

$$L\left\{u^{\alpha+n}\ v^{\beta+n}\left(1+\frac{xus_1}{2}+\frac{yvs_2}{2}\right)^n:s_1,s_2\right\}$$

$$= \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{s_1^{\alpha+n+1}\, s_2^{\beta+n+1}} Y_n^{(\alpha,\beta)}\,(x,\, y) \tag{6.4}$$

In an attempt to obtain results analogous to (6.3) and (6.4) for $Y_n^{(\alpha,\beta,\gamma)}(x,\, y, z; a, b, c)$ we define a three variable analogue of (6.1) as follows

$$L\{f\,(u, v, w): s_1, s_2, s_3\} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1 u - s_2 v - s_3 w}\, f(u, v, w)\, du\, dv\, dw \tag{6.5}$$

Now, we have

$$L\left\{u^{-\alpha-n-1}\, v^{-\beta-n-1}\, w^{-\gamma-n-1}\, Y_n^{(\alpha,\beta,\gamma)}\left(\frac{ax}{us_1}, \frac{by}{vs_2}, \frac{cz}{ws_3}; a, b, c\right): s_1, s_2, s_3\right\}$$

$$= \frac{(-1)^{n+1}\, \pi^3\, s_1^{\alpha+n}\, s_2^{\beta+n}\, s_3^{\gamma+n}}{\sin\pi\alpha\, \sin\pi\beta\, \sin\pi\gamma\, \Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}(1-x-y)^n \tag{6.6}$$

Similarly, we obtain

$$L\left\{u^{\alpha+n}\, v^{\beta+n}\, w^{\gamma+n}\left(1+\frac{xus_1}{a}+\frac{yvs_2}{b}+\frac{zws_3}{c}\right)^n : s_1, s_2, s_3\right\}$$

$$= \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}{s_1^{\alpha+n+1}\, s_2^{\beta+n+1}\, s_3^{\gamma+n+1}} Y_n^{(\alpha,\beta,\gamma)}(x,\, y, z; a, b, c) \tag{6.7}$$

VII. Generating Functions

It is easy to derive the following generating functions for $Y_n^{(\alpha,\beta,\gamma)}(x,\, y, z; a, b, c)$:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} Y_n^{(\alpha-n,\beta-n,\gamma-n)}(x,\, y, z; a, b, c) = e^t\left(1-\frac{xt}{a}\right)^{-\alpha-1}\left(1-\frac{yt}{b}\right)^{-\beta-1}\left(1-\frac{zt}{c}\right)^{-\gamma-1} \tag{7.1}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} Y_n^{(\alpha,\beta-n,\gamma-n)}(x,\, y, z; a, b, c)$$

$$= \left(1-\frac{4xt}{a}\right)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-\frac{4xt}{a}}}\right]^{\alpha}\left[1-\frac{\frac{2yt}{b}}{1+\sqrt{1-\frac{4xt}{a}}}\right]^{-\beta-1}\left[1-\frac{\frac{2zt}{c}}{1+\sqrt{1-\frac{4xt}{a}}}\right]^{-\gamma-1} e^{\frac{2t}{1+\sqrt{1-\frac{4xt}{a}}}} \tag{7.2}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} Y_n^{(\alpha-n,\beta,\gamma-n)}(x,\, y, z; a, b, c)$$

$$= \left(1-\frac{4yt}{b}\right)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-\frac{4yt}{b}}}\right]^{\beta}\left[1-\frac{\frac{2xt}{a}}{1+\sqrt{1-\frac{4yt}{b}}}\right]^{-\alpha-1}\left[1-\frac{\frac{2zt}{c}}{1+\sqrt{1-\frac{4yt}{b}}}\right]^{-\gamma-1} e^{\frac{2t}{1+\sqrt{1-\frac{4yt}{b}}}} \tag{7.3}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} Y_n^{(\alpha-n,\beta-n,\gamma)}(x,\, y, z; a, b, c)$$

$$=\left(1-\frac{4zt}{c}\right)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-\frac{4zt}{c}}}\right]^{\gamma}\left[1-\frac{\frac{2xt}{a}}{1+\sqrt{1-\frac{4zt}{c}}}\right]^{-\alpha-1}\left[1-\frac{\frac{2yt}{b}}{1+\sqrt{1-\frac{4zt}{c}}}\right]^{-\beta-1} e^{\frac{2t}{1+\sqrt{1-\frac{4zt}{c}}}} \quad (7.4)$$

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}Y_n^{(\alpha-2n,\beta-n,\gamma-n)}(x,y,z;a,b,c)$$
$$=e^{\frac{at}{a+xt}}\left(1+\frac{xt}{a}\right)^{\alpha}\left\{1-\frac{ayt}{b(a+xt)}\right\}^{-\beta-1}\left\{1-\frac{azt}{c(a+xt)}\right\}^{-\gamma-1} \quad (7.5)$$

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}Y_n^{(\alpha-n,\beta-2n,\gamma-n)}(x,y,z;a,b,c)$$
$$=e^{\frac{bt}{b+yt}}\left(1+\frac{yt}{b}\right)^{\beta}\left\{1-\frac{bxt}{a(b+yt)}\right\}^{-\alpha-1}\left\{1-\frac{bzt}{c(b+yt)}\right\}^{-\gamma-1} \quad (7.6)$$

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}Y_n^{(\alpha-n,\beta-n,\gamma-2n)}(x,y,z;a,b,c)$$
$$=e^{\frac{ct}{c+zt}}\left(1+\frac{zt}{c}\right)^{\gamma}\left\{1-\frac{cxt}{a(c+zt)}\right\}^{-\alpha-1}\left\{1-\frac{cyt}{b(c+zt)}\right\}^{-\beta-1} \quad (7.7)$$

$$\sum_{k=0}^{\infty}(-\lambda)^k\,Y_n^{(\alpha,\beta,k-n)}(x,y,z;a,b,c)=\frac{1}{1+\lambda}Y_n^{(\alpha,\beta,-n)}(x,y,z;a,b,c(1+\lambda)) \quad (7.8)$$

$$\sum_{k=0}^{\infty}(-\lambda)^k\,Y_n^{(\alpha,k-n,\gamma)}(x,y,z;a,b,c)=\frac{1}{1+\lambda}Y_n^{(\alpha,-n,\gamma)}(x,y,z;a,b(1+\lambda),c) \quad (7.9)$$

$$\sum_{k=0}^{\infty}(-\lambda)^k\,Y_n^{(k-n,\beta,\gamma)}(x,y,z;a,b,c)=\frac{1}{1+\lambda}Y_n^{(-n,\beta,\gamma)}(x,y,z;a(1+\lambda),b,c) \quad (7.10)$$

Using (3.1), we can also derive the following results :

$$\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}Y_n^{(\alpha,\beta,k-n)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}u^{\alpha+n}v^{\beta+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}J_0\left(2\sqrt{\lambda w}\right)du\,dv\,dw$$
$$(7.11)$$

$$\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}Y_n^{(\alpha,k-n,\gamma)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\alpha+n+1)\Gamma(\gamma+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}u^{\alpha+n}w^{\gamma+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}J_0\left(2\sqrt{\lambda v}\right)du\,dv\,dw$$
$$(7.12)$$

$$\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}Y_n^{(k-n,\beta,\gamma)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}u^{\beta+n}w^{\gamma+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}J_0\left(2\sqrt{\lambda u}\right)du\,dv\,dw$$
$$(7.13)$$

$$\sum_{k=0}^{\infty} (-1)^k \ \lambda^{2k} \ Y_n^{(\alpha,\beta,2k-n)}(x,y,z;a,b,c)$$
$$= \frac{1}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} \int_0^\infty \int_0^\infty \int_0^\infty u^{\alpha+n} v^{\beta+n} \left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w} \cos \lambda w \ du\, dv\, dw \tag{7.14}$$

$$\sum_{k=0}^{\infty} (-1)^k \ \lambda^{2k} \ Y_n^{(\alpha,2k-n,\gamma)}(x,y,z;a,b,c)$$
$$= \frac{1}{\Gamma(\alpha+n+1)\Gamma(\gamma+n+1)} \int_0^\infty \int_0^\infty \int_0^\infty u^{\alpha+n} w^{\gamma+n} \left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w} \cos \lambda v \ du\, dv\, dw \tag{7.15}$$

$$\sum_{k=0}^{\infty} (-1)^k \ \lambda^{2k} \ Y_n^{(2k-n,\beta,\gamma)}(x,y,z;a,b,c)$$
$$= \frac{1}{\Gamma(\beta+n+1)\Gamma(\gamma+n+1)} \int_0^\infty \int_0^\infty \int_0^\infty u^{\beta+n} w^{\gamma+n} \left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w} \cos \lambda u \ du\, dv\, dw \tag{7.16}$$

$$\sum_{k=0}^{\infty} (-1)^k \ \lambda^{2k+1} \ Y_n^{(\alpha,\beta,2k+1-n)}(x,y,z;a,b,c)$$
$$= \frac{1}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} \int_0^\infty \int_0^\infty \int_0^\infty u^{\alpha+n} v^{\beta+n} \left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w} \sin \lambda w \ du\, dv\, dw \tag{7.17}$$

$$\sum_{k=0}^{\infty} (-1)^k \ \lambda^{2k+1} \ Y_n^{(\alpha,2k+1-n,\gamma)}(x,y,z;a,b,c)$$
$$= \frac{1}{\Gamma(\alpha+n+1)\Gamma(\gamma+n+1)} \int_0^\infty \int_0^\infty \int_0^\infty u^{\alpha+n} w^{\gamma+n} \left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w} \sin \lambda v \ du\, dv\, dw \tag{7.18}$$

$$\sum_{k=0}^{\infty} (-1)^k \ \lambda^{2k+1} \ Y_n^{(2k+1-n,\beta,\gamma)}(x,y,z;a,b,c)$$
$$= \frac{1}{\Gamma(\beta+n+1)\Gamma(\gamma+n+1)} \int_0^\infty \int_0^\infty \int_0^\infty u^{\beta+n} w^{\gamma+n} \left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w} \sin \lambda u \ du\, dv\, dw \tag{7.19}$$

VIII. Double Generating Functions

The following double generating functions for $Y_n^{(\alpha,\beta,\gamma)}(x,y,z;a,b,c)$ can easily be derived by using (3.1)

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} (-\lambda)^m (-\mu)^k \ Y_n^{(m-n,k-n,\gamma)}(x,y,z;a,b,c)$$
$$= \frac{1}{(1+\lambda)(1+\mu)} \ Y_n^{(-n,-n,\gamma)}(x,y,z;a(1+\lambda),b(1+\mu),c) \tag{8.1}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} (-\lambda)^m (-\mu)^k \ Y_n^{(m-n,\beta,k-n)}(x,y,z;a,b,c)$$
$$= \frac{1}{(1+\lambda)(1+\mu)} \ Y_n^{(-n,\beta,-n)}(x,y,z;a(1+\lambda),b,c(1+\mu)) \tag{8.2}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-\lambda)^m(-\mu)^k\, Y_n^{(\alpha,\, m-n,\, k-n)}(x, y, z; a, b, c)$$

$$= \frac{1}{(1+\lambda)(1+\mu)}\; Y_n^{(\alpha,-n,-n)}(x, y, z; a, b(1+\lambda), c(1+\mu)) \tag{8.3}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\lambda)^m\;(-\mu)^k}{m!\;\; k!}\, Y_n^{(m-n,\, k-n,\gamma)}(x, y, z; a, b, c)$$

$$= \frac{1}{\Gamma(\gamma+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} w^{\gamma+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\, J_0\left(2\sqrt{\lambda u}\right)J_0\left(2\sqrt{\mu v}\right) du\, dv\, dw \tag{8.4}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\lambda)^m\;(-\mu)^k}{m!\;\; k!}\, Y_n^{(m-n,\beta,\, k-n)}(x, y, z; a, b, c)$$

$$= \frac{1}{\Gamma(\beta+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} v^{\beta+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\, J_0\left(2\sqrt{\lambda u}\right)J_0\left(2\sqrt{\mu w}\right) du\, dv\, dw \tag{8.5}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\lambda)^m\;(-\mu)^k}{m!\;\; k!}\, Y_n^{(\alpha,\, m-n,\, k-n)}(x, y, z; a, b, c)$$

$$= \frac{1}{\Gamma(\alpha+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} u^{\alpha+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\, J_0\left(2\sqrt{\lambda v}\right)J_0\left(2\sqrt{\mu w}\right) du\, dv\, dw \tag{8.6}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\;\lambda^{2m}\mu^{2k}\, Y_n^{(2m-n,\, 2k-n,\gamma)}(x, y, z; a, b, c)$$

$$= \frac{1}{\Gamma(\gamma+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} w^{\gamma+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\cos\lambda u\,\cos\mu v\;\; du\, dv\, dw \tag{8.7}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\;\lambda^{2m}\mu^{2k}\, Y_n^{(2m-n,\beta,\, 2k-n)}(x, y, z; a, b, c)$$

$$= \frac{1}{\Gamma(\beta+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} v^{\beta+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\cos\lambda u\;\cos\mu w\;\; du\, dv\, dw \tag{8.8}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\;\lambda^{2m}\mu^{2k}\, Y_n^{(\alpha,2m-n,\, 2k-n)}(x, y, z; a, b, c)$$

$$= \frac{1}{\Gamma(\alpha+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} u^{\alpha+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\cos\lambda v\;\cos\mu w\;\; du\, dv\, dw \tag{8.9}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\;\lambda^{2m+1}\mu^{2k}\, Y_n^{(2m+1-n,\, 2k-n,\gamma)}(x, y, z; a, b, c)$$

$$= \frac{1}{\Gamma(\gamma+n+1)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} w^{\gamma+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\sin\lambda u\,\cos\mu v\;\; du\, dv\, dw \tag{8.10}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\;\lambda^{2m+1}\mu^{2k}\, Y_n^{(2m+1-n,\beta,\; 2k-n)}(x, y, z; a, b, c)$$

$$=\frac{1}{\Gamma(\beta+n+1)}\int_0^\infty\int_0^\infty\int_0^\infty v^{\beta+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\sin\lambda u\cos\mu w\;du\,dv\,dw \tag{8.11}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m+1}\mu^{2k}Y_n^{(\alpha,2m+1-n,2k-n)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\alpha+n+1)}\int_0^\infty\int_0^\infty\int_0^\infty u^{\alpha+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\sin\lambda v\cos\mu w\;du\,dv\,dw \tag{8.12}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m}\mu^{2k+1}Y_n^{(2m-n,2k+1-n,\gamma)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\gamma+n+1)}\int_0^\infty\int_0^\infty\int_0^\infty w^{\gamma+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\cos\lambda u\sin\mu v\;du\,dv\,dw \tag{8.13}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m}\mu^{2k+1}Y_n^{(2m-n,\beta,2k+1-n)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\beta+n+1)}\int_0^\infty\int_0^\infty\int_0^\infty v^{\beta+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\cos\lambda u\sin\mu w\;du\,dv\,dw \tag{8.14}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m}\mu^{2k+1}Y_n^{(\alpha,2m-n,2k+1-n)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\alpha+n+1)}\int_0^\infty\int_0^\infty\int_0^\infty u^{\alpha+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\cos\lambda v\sin\mu w\;du\,dv\,dw \tag{8.15}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m+1}\mu^{2k+1}Y_n^{(2m+1-n,2k+1-n,\gamma)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\gamma+n+1)}\int_0^\infty\int_0^\infty\int_0^\infty w^{\gamma+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\sin\lambda u\sin\mu v\;du\,dv\,dw \tag{8.16}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m+1}\mu^{2k+1}Y_n^{(2m+1-n,\beta,2k+1-n)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\beta+n+1)}\int_0^\infty\int_0^\infty\int_0^\infty v^{\beta+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\sin\lambda u\sin\mu w\;du\,dv\,dw \tag{8.17}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m+1}\mu^{2k+1}Y_n^{(\alpha,2m+1-n,2k+1-n)}(x,y,z;a,b,c)$$
$$=\frac{1}{\Gamma(\alpha+n+1)}\int_0^\infty\int_0^\infty\int_0^\infty u^{\alpha+n}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\sin\lambda v\sin\mu w\;du\,dv\,dw \tag{8.18}$$

IX. Triple Generating Functions

The following triple generating functions can easily be obtained by using (3.1):

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-\lambda)^m(-\mu)^k(-\eta)^j\,Y_n^{(m-n,\,k-n,\,j-n)}(x,y,z;a,b,c)$$

$$=\frac{1}{(1+\lambda)(1+\mu)(1+\eta)}Y_n^{(-n,\,-n,\,-n)}(x,y,z;a(1+\lambda),b(1+\mu),c(1+\eta)) \qquad (9.1)$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\lambda)^m(-\mu)^k(-\eta)^j}{m!\,k!\,j!}Y_n^{(m-n,\,k-n,\,j-n)}(x,y,z;a,b,c)$$

$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}J_0\left(2\sqrt{\lambda u}\right)J_0\left(2\sqrt{\mu v}\right)J_0\left(2\sqrt{\eta w}\right)du\,dv\,dw \qquad (9.2)$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{m+k+j}\lambda^{2m}\mu^{2k}\eta^{2j}\,Y_n^{(2m-n,\,2k-n,\,2j-n)}(x,y,z;a,b,c)$$

$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\cos\lambda u\,\cos\mu v\,\cos\eta w\,du\,dv\,dw \qquad (9.3)$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{m+k+j}\lambda^{2m+1}\mu^{2k+1}\eta^{2j+1}\,Y_n^{(2m+1-n,\,2k+1-n,\,2j+1-n)}(x,y,z;a,b,c)$$

$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\sin\lambda u\,\sin\mu v\,\sin\eta w\,du\,dv\,dw \qquad (9.4)$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{m+k+j}\lambda^{2m+1}\mu^{2k+1}\eta^{2j}\,Y_n^{(2m+1-n,\,2k+1-n,\,2j-n)}(x,y,z;a,b,c)$$

$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\sin\lambda u\,\sin\mu v\,\cos\eta w\,du\,dv\,dw \qquad (9.5)$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{m+k+j}\lambda^{2m}\mu^{2k}\eta^{2j+1}\,Y_n^{(2m-n,\,2k-n,\,2j+1-n)}(x,y,z;a,b,c)$$

$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left(1+\frac{xu}{a}+\frac{yv}{b}+\frac{zw}{c}\right)^n e^{-u-v-w}\cos\lambda u\,\cos\mu v\,\sin\eta w\,du\,dv\,dw \qquad (9.6)$$

X. Bessel Polynomials Of M-Variables

The Bessel polynomials of m-variables $Y_n^{(\alpha_1,\alpha_2,----,\alpha_m)}(x_1,x_2,----,x_m;a_1,a_2,----,a_m)$ can be defined as follows:

$$Y_n^{(\alpha_1,\alpha_2,----,\alpha_m)}(x_1,x_2,----,x_m;a_1,a_2,----,a_m)$$

$$\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\sum_{r_3=0}^{n-r_1-r_2}----\sum_{r_m=0}^{n-r_1-r_2-----r_{m-1}}\frac{(-n)_{r_1+r_2+---+r_m}\prod_{i=0}^{m}(\alpha_i+n+1)_j}{\prod_{i=0}^{m}r_i\,!}\prod_{i=0}^{m}\left(-\frac{x_i}{a_i}\right) \qquad (10.1)$$

All the results of this paper can be extended for this m-variable Bessel polynomials. The only hinderance in their study is the representation of results in hypergeometric functions of m-variables.

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