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International Journal Of Modern Engineering Research (IJMER)

Some Operation Equation and Applications

Ning Chen[1], Jiqian Chen[2]

[1](College of Scienc , Southwest University of Science and Technology, Mianyang 621010, PR China)

[2](College of Scienc , Southwest University of Science and Technology, Mianyang 621010, PR China

ABSTRACT: *In this paper, we give several new fixed point theorems to extend results [3]-[4] ,and we apply the effective modification of He's variation iteration method to solve some nonlinear and linear equations are proceed to examine some a class of integral-differential equations and some partial differential equation, to illustrate the effectiveness and convenience of this method(see[7]). Finally we have also discussed Berge type equation with exact solution.*

Keywords: *Semi-closed 1-set-contractive operator, Topology degree, Modified variation iteration method, Integral-differential equation, Partial Differential Equation*

I. Introduction And Preliminaries

The introduction of the paper should explain the nature of the problem, previous work, purpose, and the contribution of the paper. The contents of each section may be provided to understand easily about the paper. In recent years, the fixed point theory and application has rapidly development.

The topological degree theory and fixed point index theory play an important role in the study of fixed points for various classes of nonlinear operators in Banach spaces (see [1-2],[3]and[4],[7],etc.). The new conclusions of main theorems: Theorem 2.1 and theorem 2.4, theorem 2.6 and theorem 2.7, theorem 3.1 and theorem 4.1. Next, the effective methods of variation iterative method to combine some integral-differential equation and partial differential equations with He's iterative method. First, we need following some definitions and conclusion (see [3]).

Let E be a real Banach space, Ω a bounded open subset of E and θ the zero element of E If $A:\overline{\Omega}\to E$ is a complete continuous operator, we have some well known theorems for needing Lemma 1.1 as follows (see [3-4]).

Lemma 1.1 (see Corollary 2.1 [3]) Let E be a real Banach Space, Ω is a bounded open subset of E and $\theta\in\Omega$. If $A:\overline{\Omega}\to E$ is a semi-closed 1-set -contractive operator such that satisfies the Leray-Schauder boundary condition $Ax\neq tx$, for all $x\in\partial\Omega$ and $t\geq 1$, then $\deg(I-A,\Omega,\theta)=1$, and so A has a fixed point in Ω.

II. Several fixed point theorems

In recent years, some new types fixed point theory and application to study the differential-integral equations in the physic and mechanics fields. Therefore, some application has rapidly development. First, we extended some results as follows. For convenience, we give out following Theorem 2.1.

Theorem2.1 Let E be a real Banach Space, Ω is a bounded open subset of E and $\theta\in\Omega$. If $A:\overline{\Omega}\to E$ is a semi-closed 1-set-contractive operator such that

$$\|Ax+(m^2+1)x\|^{n(\alpha+\beta)+\gamma}\leq\|Ax+mx\|^{n\alpha+\gamma}\|Ax\|^{n\beta}+\|x\|^{n(\alpha+\beta)+\gamma}\text{, for all } x\in\partial\Omega. \tag{2.1}$$

(where $\alpha>1,\beta\geq 0,\gamma\geq 0$, $m,n-$positive integer)

Then $\deg(I-A,\Omega,\theta)=1$, if A has no fixed points on $\partial\Omega$ and so A has a fixed point in $\overline{\Omega}$.

Proof By lemma 1.1, we can prove theorem 2.1 .Suppose that A has no fixed point on $\partial\Omega$ .Then assume it is not true, there exists $x_0\in\partial\Omega,\mu_0\geq 1$ such that $Ax_0=\mu_0 x_0$.

It is easy to see that $\mu_0>1$. Now, consider the function defined by

$f(t)=(t+(m^2+1))^{n(\alpha+\beta)+\gamma}-(t+m)^{n(\alpha+\beta)+\gamma}-1,$ for any $t\geq 1$.

Since $f'(t)=(n\alpha+n\beta+\gamma)[(t+(m^2+1))^{n(\alpha+\beta)+\gamma-1}-(t+m)^{n(\alpha+\beta)+\gamma-1}]>0,\ t\geq 1$ and $f'(t)>0$.

So, $f(t)$ is a strictly increasing function in $[1,+\infty)$, and $f(t)>f(1)$ for $t>1$. Thus,

Consequently, noting that $\|x_0\|\neq 0,\ \mu_0>1$, we have

$$\|Ax_0+(m^2+1)x_0\|^{n(\alpha+\beta)+\gamma}>[(\mu_0+m)^{n(\alpha+\beta)+\gamma}+1]\|x_0\|^{n(\alpha+\beta)+\gamma}$$
$$=\|Ax_0+mx_0\|^{n\alpha+\gamma}\|Ax_0\|^{n\beta}+\|x_0\|^{n(\alpha+\beta)+\gamma},$$

which contradicts (2.1), and so the condition $(L-S)$ is satisfied. Therefore, it follows from lemma 1.1 that the conclusion of theorem 2.1 holds.

Corollary 2.2. Let $m=1$, we get theorem 2.3 [3].

Corollary2.3. If $\|Ax+(m^2+1)x\|^{n(\alpha+\beta)+\gamma}\le\|Ax+mx\|^{n\alpha+\gamma}\|Ax\|^{n\beta}$, then $\deg(I-A,\Omega,\theta)=1$ by similar proof method.

In fact, $\|Ax+mx\|^{n\alpha+\gamma}\|Ax\|^{n\beta}\le\|Ax+mx\|^{n\alpha+\gamma}\|Ax\|^{n\beta}+\|x\|^{n(\alpha+\beta)+\gamma}$,it satisfies condition of theorem 2.1.

In the same reason, we extend some theorem [3] as follows.

Theorem 2.4 Let E be a real Banach Space, Ω is a bounded open subset of E and $\theta\in\Omega$. If $A:\overline{\Omega}\to E$ is a semi-closed 1-set-contractive operator such that satisfies condition:

$$\|Ax-kx\|^3+k\|Ax+x\|^3\neq(k+1)[\|Ax\|^3+3\|Ax\|\cdot\|x\|^2+(1-k^2)\|x\|^2)],\ \text{for all } x\in\partial\Omega. \tag{2.4}$$

Then deg $(I-A,\Omega,\theta)=1$

Proof Similar as above stating, we shall prove that the L-S condition is satisfied. That is there exists $x_0\in\partial\Omega,\mu_0\ge1$ such that $Ax_0=\mu_0x_0$, and $\mu_0>1$. Now, consider it by (2.4), we have that

$$(\mu_0-k)^3+k(\mu_0+1)^3\neq(k+1)(\mu_0^3+k^3\mu_0).$$

This is a contradiction. By lemma 1.1, then deg $(I-A,\Omega,\theta)=1$ that A has a fixed point in $\overline{\Omega}$. We complete this proof.

Theorem2.5.Let E,Ω,A be the same as theorem 2.7. Moreover, if substituting (2.4) into that inequality:

$$\|Ax-x\|^4+\|Ax+x\|^4\neq2[\|Ax\|^4+6\|Ax\|^2\|x\|^2+\|x\|^4),\ \text{for all } x\in\partial\Omega. \tag{2.5}$$

Then deg $(I-A,\Omega,\theta)=1$, then the A has at least one fixed point in $\overline{\Omega}$.

Proof Similar as the proof of theorem2.4. If we suppose that A has no fixed point on $\partial\Omega$. Then we shall prove that the L-S condition is satisfied. We assume it is not true**,** there exists $x_0\in\partial\Omega,\mu_0\ge1$such that $Ax_0=\mu_0x_0$, and $\mu_0>1$. Now, consider it by (2.5), we have that $(\mu_0-1)^4+(\mu_0+1)^4=2(\mu_0^4+6\mu_0{}^2+1)$.

By (2.5) we have that $(\mu_0-1)^4+(\mu_0+1)^4\neq2(\mu_0^4+6\mu_0{}^2+1)$. Thus, this is a contradiction, then by lemma 1.1 that we get the conclusions of theorem 2.5.

Therefore, we shall consider some higher degree case as follow (Omit similar proof).

Theorem 2.6 Let E be a real Banach Space, Ω is a bounded open subset of E and $\theta\in\Omega$. If $A:\overline{\Omega}\to E$ is a semi-closed 1-set-contractive operator such that satisfies condition:

$$\|Ax-x\|^{2n+1}+\|Ax+x\|^{2n+1}\neq2[\|Ax\|^{2n}+C_{2n}^2\|Ax\|^{2n-2}\|x\|^2$$
$$+C_{2n}^4\|Ax\|^{2n-4}\|x\|^4+\cdots+C_{2n}^{2n-4}\|Ax\|^4\cdot\|x\|^{2n-4}+C_{2n}^{2n-2}\|Ax\|^2\cdot\|x\|^{2n-2}+\|x\|^{2n}],\ \text{for every } x\in\partial\Omega \tag{2.6}$$

Then deg $(I-A,\Omega,\theta)=1$, then if A has no fixed points on $\partial\Omega$ and so A has at least one fixed point in $\overline{\Omega}$.

Theorem 2.7 Let E be a real Banach Space, Ω is a bounded open subset of E and $\theta\in\Omega$. If $A:\overline{\Omega}\to E$ is a semi-closed 1-set-contractive operator such that satisfies condition:

$$\|Ax-x\|^{2n}+\|Ax+x\|^{2n}\neq2[\|Ax\|^{2n}+C_{2n}^2\|Ax\|^{2n-2}\|x\|^2$$
$$+C_{2n}^4\|Ax\|^{2n-4}\|x\|^4+\cdots+C_{2n}^{2n-4}\|Ax\|^4\cdot\|x\|^{2n-4}+C_{2n}^{2n-2}\|Ax\|^2\cdot\|x\|^{2n-2}+\|x\|^{2n}]\ \text{for all } x\in\partial\Omega. \tag{2.7}$$

Then deg $(I-A,\Omega,\theta)=1$, then if A has no fixed points on $\partial\Omega$ and so A has at least one fixed point in $\overline{\Omega}$ (We omit the similar proof of above Theorems).

III. Some Notes For Altman Type Inequality

It is well known we may extend that boundary condition inequality as bellow case:

$$\|Ax-mx\|^2\ge\|Ax\|^2-\|mx\|^2\ \text{for all } x\in\partial\Omega, m>0. \tag{3.1}$$

then the semi-closed 1-set-contractive operator A must have fixed point in $\bar{\Omega}$.

In fact, there exists $x_0 \in \partial\Omega, \mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$, and $\mu_0 > 1$. By (3.1), we have that $(\mu_0 - m)^2 \geq \mu_0^2 - m^2, 2m^2 - 2\mu_0 m \geq 0.$

Then $\Delta = \mu_9^2 > 1 > 0$, this is a contradiction.

By Lemma 1.1, we obtain deg $(I - A, \Omega, \theta) = 1,$ then operator A must have fixed point in $\bar{\Omega}$.

We can extend this Altman's inequality into the determinant type form (also see [3]-[5], [6] etc):

$$\|Ax - mx\|^2 \geq \|Ax\|^2 - \|mx\|^2 = \begin{vmatrix} \|A(x)\|, \|mx\| \\ \|mx\|, \|A(x)\| \end{vmatrix} = D_2$$

that we consider these general case: D_n for n-order determinant. Similar Corollary (2) satisfy condition bellow: $\|Ax\|^2 + \|Ax - B_i x\|^2 \leq \|Ax - mx\|^2 + \|mx\|^2 (i = 1,2,\cdots,l)$. Then if these semi-closed operators $B_1, B_2, \cdots, B_l$ have not common fixed point each other, then S have at least l numbers fixed points. Now, we write n-order determinant type form

$$D_n = \begin{vmatrix} \|Ax + x\| & \|x\| & \cdots & \|x\| \\ \|x\| & \|Ax + x\| & \cdots & \|x\| \\ \vdots & \ddots & \ddots & \vdots \\ \|x\| & \|x\| & \cdots & \|Ax + x\| \end{vmatrix}.$$

Then by simple calculation, $D_n = (\|Ax + x\| + (n-1)\|x\|)(\|Ax + x\| - \|x\|)^{n-1}.$

Moreover, in the similar discussion along this direction, we extend Corollary 2.6 in [3] with as following Theorem 3.1

Theorem 3.1 Let E, Ω, A be the same as in lemma 1.1.Moreover,if there exists n-positive integer such that $(n \geq 2)$,

$$\|Ax\|^{2(n+1)} \geq D_n D_{n+2}, \quad \text{for all } x \in \partial\Omega, \tag{3.2}$$

Then $\deg(I - A, \Omega, \theta) = 1,$ if A has no fixed points on $\partial\Omega,$ and so A has at least one fixed point in $\bar{\Omega}$.

Proof Noting the determinant by D_n , $D_{n+2} = (\|Ax + x\| + (n+1)\|x\|)(\|Ax + x\| - \|x\|)^{n+1}.$

By (3.2), we have

$$\|Ax\|^{2(n+1)} \geq (\|Ax + x\| + (n-1)\|x\|)(\|Ax + x\| + (n+1)\|x\|)(\|Ax + x\| - \|x\|)^{2n}.$$

In fact, there exists $x_0 \in \partial\Omega, \mu_0 \geq 1$ x_0 $x_0 \in \partial\Omega, \mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that $\mu_0 > 1$.

Now, we have the inequality

$$\mu_0^{2n+2} \geq [(\mu_0 + (n-1))][\mu_0 + (n+1)]\mu_0^{2n} > \mu_0^{2n+2}.$$

Hence, $(\|Ax_0 + x_0\| + (n-1)\|x_0\|)(\|Ax_0 + x_0\| + (n+1)\|x_0\|)(\|Ax_0 + x_0\| - \|x_0\|)^{2n} > \|Ax_0\|^{2n+2},$

Which is a contradiction to (3.2), and so the boundary condition of Leray-Schauder condition is satisfied. Therefore, it follows from lemma 1.1 that the conclusion of Theorem 3.1 holds.

Theorem 3.2 Let be the same as in lemma 1.1.Moreover, if there exists $n-$ positive integer such that $(n \geq 2)$,

$$D_{n+1}{}^2 \neq D_n D_{n+2}, \quad \text{for all } x \in \partial\Omega, \tag{3.3}$$

Then $\deg(I - A, \Omega, \theta) = 1,$ if A has no fixed points on $\partial\Omega,$ and so A has at least one fixed point in $\bar{\Omega}$. (Omit this similar proof).

Theorem 3.3 Let E, Ω, A be the same as in lemma 1.And, if there exists $n-$ positive integer such that $(n \geq 2)$.

$$\|Ax - x\|^{3(n+1)} \geq D_n D_{n+1} D_{n+2}, \quad \text{for all } x \in \partial\Omega, \tag{3.4}$$

Then $\deg(I - A, \Omega, \theta) = 1,$ if A has no fixed points on $\partial\Omega,$ and so A has at least one fixed point in $\bar{\Omega}$.

Proof Notice the determinant D_n , then $D_{n+1} = (\|Ax + x\| + n\|x\|)(\|Ax + x\| - \|x\|)^n$, and $D_{n+2} = (\|Ax + x\| + (n+1)\|x\|)(\|Ax + x\| - \|x\|)^{n+1},$ similar as in above way that we omit the similar proof of Theorem 3.2. In fact, on the contrary, there exists $x_0 \in \partial\Omega, \mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that

$\mu_0 > 1$. It is well know by (3.4), we easy obtain that there is a contraction for this inequality. So A has at least one fixed point in $\bar{\Omega}$.
Theorem 3.4 Suppose that same as Theorem 3.3, satisfy follows form:

$$\|Ax - x\|^{2(n+3)} \geq D_n D_{n+1} D_{n+2} D_{n+3}, \text{ for all } x \in \partial\Omega, \tag{3.5}$$

Then $\deg(I - A, \Omega, \theta) = 1$, if A has no fixed points on $\partial\Omega$ and so A has at least one fixed point in $\bar{\Omega}$. (Omit this similar proof of Theorem 3.4).
Remark Notes that new case of non-symmetry form with D_n for n-order determinant are given with more conclusions.

IV. solution of integral equation by vim

To ensure a high-quality product, diagrams and lettering MUST be either computer-drafted or drawn using India ink. Recently, the variation iteration method (VIM) has been favorably applied to some various kinds of nonlinear problems, for example, fractional differential equations, nonlinear differential equations, nonlinear thermo-elasticity, nonlinear wave equations.

They have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics, and engineer sciences.

In this section, we apply the variation iteration method (simple writing VIM) to Integral-differential equations bellow (see [3] and [4-6] etc.). To illustrate the basic idea of the method, we consider:

$$L[u(t)] + N[u(t)] = g(t),$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a continuous function.
The basic character of the method is to construct functional for the system, which reads:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s) Lu\left(L\bar{u}_n + Nu_n - g(s)\right) ds,$$

where λ is the Lagrange multiplier which can be identified optimally via variation theory, u_n is the nth approximate solution, and $\bar{u}_n$ denotes a restricted variation, $\delta\bar{u}_n = 0$. There is an iterative formula:

$$u_{n+1}(x) = f(x) + \lambda \int_a^b k(x,t) u_n(t) dt$$

of this integral equation.
Theorem 4.1 ([4, theorem 4.1]).Consider the iteration scheme $u_0(x) = f(x)$, and the

$$u_{n+1}(x) = f(x) + \lambda \int_a^b k(x,t) u_n(t) dt. \tag{4.1a}$$

Now, for $n = 0,1,2,\cdots$, to construct a sequence of successive iterations that for the $\{u_n(t)\}$ to the solution of integral equation (4.1). In addition, we assume that $\int_a^b \int_a^b k^2(x,t) dxdt = B^2 < \infty$, and assume that $f(x) \in L^2_{[a,b]}$, if $|\lambda| < 1/B$, then the above iteration converges in the norm of $L^2_{[a,b]}$ to the solution of integral equation (4.1).

Corollary4.2 If $k(x,t) = \sum_{i=1}^{m} a_i(x) b_i(t)$, and $\int_a^b \int_a^b k^2(x,t) dxdt = B^2 < \infty$, then assume $f(x) \in L^2_{[a,b]}$, if $|\lambda| < 1/B$, the above iteration converges in the norm of $L^2_{[a,b]}$ to the solution of integral equation (4.1).

Corollary4.3 If $k(x,t) = k_1(x,t) + k_2(x,t)$, and $\int_a^b \int_a^b k^2(x,t) dxdt = B^2 < \infty$, then assume $f(x) \in L^2_{[a,b]}$, if $|\lambda| < 1/B$, the above iteration converges in the norm of $L^2_{[a,b]}$ to the solution of integral equation (4.1).

Example4.1. Consider that integral equation

$$u(x) = x^\alpha + x^3 + x^2 + \lambda \int_0^1 (xt + t) u(t) dt. \tag{4.2}$$

Where $u_0(x) = x^\alpha + x^3 + x^2, (0 < \alpha < 1)$, let $\gamma = (\alpha + 2)^{-1} + 5^{-1} + 4^{-1}$ and that

$$u_{n+1}(x) = x^{\alpha} + x^3 + x^2 + \lambda \int_0^1 (xt+t) u_n(t) dt \tag{4.2a}$$

Where $\int_a^b \int_a^b k^2(x,t) dxdt = \int_0^1 \int_0^1 (xt+t)^2 dxdt = 9^{-1} + 3^{-1} + 3^{-1} = 7/9 = B^2 < \infty$, if $|\lambda| < 3/\sqrt{7}$, then the iterative (4.1a) is convergent to the solution of equation (4.2). Substituting $u_0(x)$ in to (4.2a), it is that (writing $P_{\alpha}(x) = x^{\alpha} + x^3 + x^2$)

$$u_1(x) = x^{\alpha} + x^3 + x^2 + \lambda \int_0^1 (xt+t) u_0(t) dt = x^{\alpha} + x^3 + x^2 + \lambda\gamma(x+1),$$

$$u_2(x) = P_{\alpha}(x) + \lambda \int_0^1 (xt+t) u_1(t) dt = P_{\alpha}(x) + \lambda \int_0^1 (x+1)t \left(t^{\alpha} + t^3 + t^2 + \lambda\gamma(t+1)\right) dt$$

$$= P_{\alpha}(x) + 2\gamma(x+1)\left(\lambda + \lambda^2/4\right)$$

$$u_3(x) = P_{\alpha}(x) + 2\gamma(x+1)\left(\lambda + \lambda^2/4\right) + \lambda \int_0^1 (x+1)t \cdot u_2(t) dt = P_{\alpha}(x) + 2\gamma(x+1)\left(\lambda + (\lambda/3)^2 + (\lambda/3)^3\right)$$

By inductively, $u_{n+1}(x) = P_{\alpha}(x) + \lambda\gamma x\left(1 + \lambda/3 + (\lambda/3)^2 + \cdots + (\lambda/3)^n\right)$

Then the solution: $u(x) = \lim_{n\to\infty} u_n(x) = P_{\alpha}(x) + 3\lambda\gamma x/(3-\lambda)$.(See Fig 1and Fig 2)

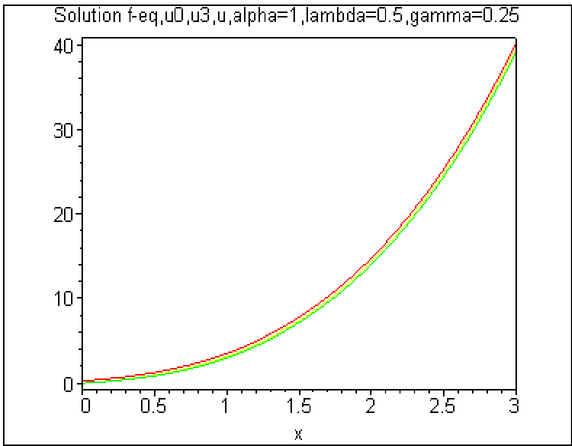

Fig 1

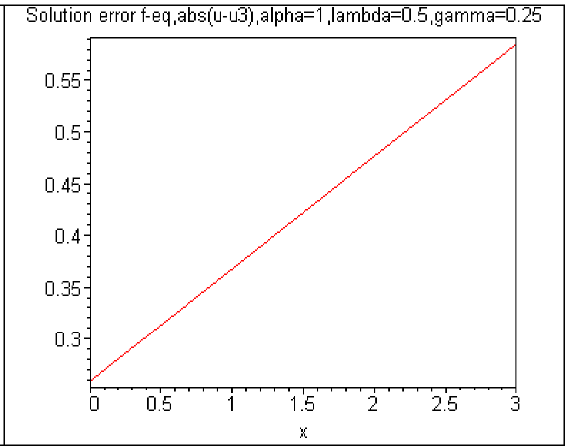

Fig 2

Example 4.2 Consider that integral equation (positive integer $k \geq 1$)

$$u(x) = x^{\alpha} + x^{k+2} + x^k + \lambda \int_0^1 (x^k t) u(t) dt \tag{4.3}$$

Where $u(0) = 0$, and $u_0(x) = x^{\alpha} + x^{k+2} + x^k \ (0 < \alpha < 1)$

From that $$u_n(x) = x^{\alpha} + x^{k+2} + x^k + \lambda \int_0^1 (x^k t) u_{n-1}(t) dt$$

By theorem 4.1 and by simple computation, we obtain again that

$$\int_a^b \int_a^b k^2(x,t) dxdt = \int_0^1 \int_0^1 (x^k t)^2 dxdt = 1/3(2k+1) = B^2 < \infty,$$

then if $|\lambda| < \sqrt{3(2k+1)}$, then iterative

$$u_n(x) = x^{\alpha} + x^{k+2} + x^k + \lambda \int_0^1 (x^k t) u_{n-1}(t) dt,$$

which is convergent the solution of integral equation (4.3). We may omit the detail calculating (Let $\gamma = (\alpha+2)^{-1}$ $+(k+4)^{-1} + (k+2)^{-1}$).

$$u_{n+1}(x) = x^{\alpha} + x^{k+2} + x^k + \lambda x^k \gamma\left(1 + (\lambda/(k+2)) + (\lambda/(k+2))^2 + \cdots + (\lambda/(k+2))^n\right).$$

$$u_n(x) = x^{\alpha} + x^{k+2} + x^k + \lambda \int_0^1 (x^k t) u_{n-1}(t) dt,$$

which is convergent the solution of integral equation (4.3). We may omit the detail calculating (Let $\gamma = (\alpha+2)^{-1} + (k+4)^{-1} + (k+2)^{-1}$).

$$u_{n+1}(x) = x^{\alpha} + x^{k+2} + x^{k} + \lambda x^{k}\gamma\left(1+\left(\lambda/(k+2)\right)+\left(\lambda/(k+2)\right)^{2}+\cdots+\left(\lambda/(k+2)\right)^{n}\right).$$

The solution $u_{n+1}(x) = x^{\alpha} + x^{k+2} + x^{k} + \lambda x^{k}\gamma(k+2)/(k+2-\lambda).$

Theorem 4.2 (see theorem 3 in [5]) Let D be a bounded open convex subset in a real Banach space X and $\theta \in D$; Suppose that $A:\overline{D}\to E$ is a semi-closed 1-set-contractive operator and satisfies the following condition:

$$\|Ax - x_0\| \le \|x - x_0\| \text{ for every } x \in \partial D \text{ and } x_0 \in D. \tag{4.4}$$

Then the operator equation $Ax = x$ has a solution in D (omit the proof)

To illustrate the application of the obtained results, we consider the examples.

Example 4.3 Similar as example 1 , we consider integral equation:

$$\int_0^x \left((1/7)^{-1}\sin|t| + (1/11)^{-1}\cos|t|\right)dt - x + 2.1 = 0,\ \forall x \in [-\pi,\pi] \tag{4.5}$$

It is easy to prove that this equation has a solution in $[-\pi,\pi]$.

In fact, let $Ax = \int_0^x \left((1/7)\sin|t| + (1/11)\cos|t|\right)dt + 2.1.\ \forall x \in [-\pi,\pi]$, and that $D = [-\pi,\pi], \partial D : x = \pm\pi.$ We write $\|y\| = |y|$, for every $y \in R.$ Thus, we have

$$|A(-\pi) - 2.1| = \left|\int_0^{-\pi}\left((1/7)^{-1}\sin|t| + (1/11)^{-1}\cos|t|\right)dt\right| \le \int_{-\pi}^{0}\left((1/7)^{-1} + (1/11)^{-1}\right)dt = 18\pi/77 < |-\pi - 2.1| = \pi + 2.1$$

And

$$|A(\pi) - 2.1| = \left|\int_0^{\pi}((1/7)^{-1}\sin|t| + (1/11)^{-1}\cos|t|)dt\right| = 18\pi/77 < |\pi - 2.1| = \pi - 2.1.$$

It follows that $|Ax - 2.1| \le |x - 2.1|$, for every $x \in \partial D$. Meanwhile, A is semi-closed 1-set-contractive operator similar example 1 by theorem 4.2 that we obtain the $Ax = x$ has a solution in $[-\pi,\pi]$. That is, Eq. (4.5) has a solution in $[-\pi,\pi]$.

V. Effective Modification of He's variation iteration

In this section, we apply the effective modification method of He's VIM to solve some integral-differential equations [7]. In [7] by the variation iteration method (VIM) simulate the system of this form

$$Lu + Ru + Nu = g(x).$$

To illustrate its basic idea of the method .we consider the following general nonlinear system

$$Lu + Ru + Nu = g(x), \tag{5.1}$$

Lu shows the highest derivative term and it is assumed easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L_x^{-1} to both sides of equation (*), then we obtain $u = f - L_x^{-1}[Ru] - L_x^{-1}[Nu]$.

The variation iteration method (VIM) proposed by Ji-Huan He (see [5], [7] has recently been intensively studied by scientists and engineers. the references cited therein) is one of the methods which have received much concern .It is based on the Lagrange multiplier and it merits of simplicity and easy execution. Unlike the traditional numerical methods. Along the direction and technique in [9], we may get more examples bellow.

We notice that an effective iterative method and some examples

Example5.1 Consider the following integral-differential equation

$$u'''(x) = e^{\alpha x} - (4/3)x + \lambda\int_0^1 xt\cdot u(t)dt,$$

Where $u(0) = 1, u'(0) = 1 + \alpha, u''(0) = \alpha^2$ with exact solution $u(x) = x + e^{\alpha x}.$

In fact, we check $f_0(x) = x + e^{\alpha x}, f_1(x) = -7x/4,$ to divide f in tow parts for $f(x) = f_0(x) + f_1(x)$, and writing $\delta = \left(1/\alpha - 1/\alpha^{-2}\right)e^{\alpha} + 1/\alpha^{-2}$.

$$L_x^{-1}\left(\int_0^1 xtu_0(t)dt\right) = L_x^{-1}\left(x\int_0^1 t(t + e^{\alpha t})dt\right) = L_x^{-1}\left(x/3 + \delta x\right) = \int_0^1\left((x/18) + (\delta x^3/6)\right)dx$$

$$=\left((1+3\delta)/18\right)\int_0^x x^3dx=\left(1+3\delta\right)x^4/72.$$

By $u_{n+1}(x)=x+e^{\alpha x}-(1/4)\left((1+3\delta)/18\right)x^4+L_x^{-1}\left(\int_0^1 xtu_n(t)dt\right)$. Then, we have

$$u_1(x)=x+e^{\alpha x}-\left((1+3\delta)/72\right)x^4+L_x^{-1}\left(\int_0^1 xtu_0(t)dt\right)=x+e^{\alpha x},$$

$$u_2(x)=x+e^{\alpha x}-\left((1+3\delta)/72\right)x^4+L_x^{-1}\left(\int_0^1 xtu_1(t)dt\right)=x+e^{\alpha t},\cdots,$$ and we have that

$u_n(x)=x+e^{\alpha x}$ Therefore, this is a closed form $u(x)=x+e^{\alpha x}$, shows that the method is a very convenient and only one iterative leads to the exact solution.

Example5.2 Consider the following integral-differential equation

$$u^{(6)}(x)=e^{\alpha x}-(4/3)x+\int_0^1 xtu(t)dt, \tag{5.2}$$

where $u(0)=1, u'(0)=1+\alpha, u''(0)=\alpha^2, u^{(3)}(0)=\alpha^3, u^{(4)}(0)=\alpha^4, u^{(5)}(0)=\alpha^5$

In similar example1, we easy have it.(We may omit it)

where $L_x^{-1}(\cdot)=\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x(\cdot)dtdtdtdtdtdt$, $u_0(x)=f_0(x)=x+e^{\alpha x}$,

$f_1(x)=f(x)-f_0(x)=(e^{\alpha x}-(4/3)x)-(x+e^{\alpha x})=-7x/3$, $f(x)=f_0(x)+f_1(x)$,

and writing $\delta=\left(1/\alpha-1/\alpha^{-2}\right)e^{\alpha}+1/\alpha^{-2}$.

$$L_x^{-1}\left(\int_0^1 xtu_0(t)dt\right)=L_x^{-1}\left(x\int_0^1 t(t+e^{\alpha t})dt\right)=L_x^{-1}\left(x/3+\delta x\right)$$

$$=\int_0^x\int_0^x\int_0^x\left(\int_0^x\left((x^3/18)+(\delta x^3/6)\right)dx\right)dxdxdx,$$

$$=\left((1+3\delta)/18\right)\int_0^x\int_0^x\int_0^x\left(x^4/4\right)dx=\left((1+3\delta)/18\right)x^7/7!.$$

$$u_1(x)=x+e^{\alpha x}-\left((1+3\delta)/3\right)x^7/7!+L_x^{-1}\left(\int_0^1 xtu_0(t)dt\right)=x+e^{\alpha x},\cdots,$$

and so on,

$$u_{n+1}(x)=x+e^{x}-\left((1+3\delta)/3\right)x^7/7!+L_x^{-1}\left(\int_0^1 xtu_n(t)dt\right)=x+e^{\alpha x}.$$

By simple operations, we have that

$$u_0(x)=x+e^{\alpha x}, u_1(x)=x+e^{\alpha x},\cdots,u_n(x)=x+e^{\alpha x}, n\geq 1.$$

Therefore, the exact solution in a closed form $u(x)=x+e^{\alpha x}$, shows that the method is a very convenient. The exact solution $u(x)=x+e^{x}$ is only one iterative. (See Fig 3, $\theta=0.75$ and $\theta=1.25$).

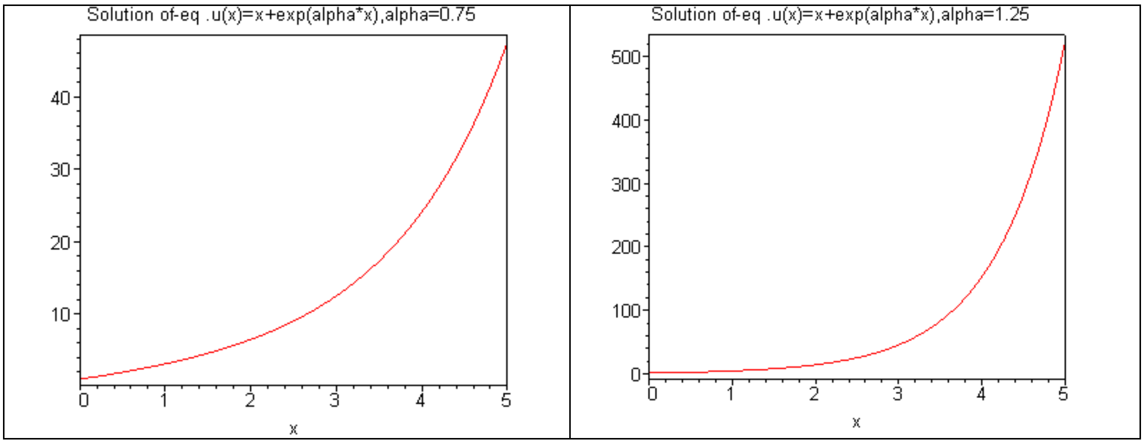

Fig 3

VI. Some Notes of Burger's Equation

We shall consider the exact and numerical solutions for Burger's equation ,which has attracted much attention .Solving this equation has been an interesting tasks for mathematicians

VI.1 One-dimensional Burger's equation

. Consider the following one-dimension Berger's equation with initial and boundary conditions:

Example 6.1 Consider the following equation (similar as example 1 [15]):

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = 3x, t > 0. \end{cases} \tag{6.0}$$

According the direction of [2] and method for this example, then it can be written as by iterative formula (6) [2]:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(\eta)(\frac{\partial u_n}{\partial \xi}(x,\xi) + u_n\frac{\partial u_n}{\partial x}(x,\xi) - v\frac{\partial^2 u_n}{\partial x^2})d\xi. \tag{6.1}$$

Starting with $u_0(x,t) = 3x.$

The following reads can be derived from iterative formula (6.1)

$$u_1(x,t) = 3x - 6xt,$$

$$u_2(x,t) = 3x - 6xt + 12xt^2 - 24xt^3,$$

$$u_3(x,t) = 3x - 6xt + 12xt^2 - 24xt^3 + 48xt^4 - 96xt^5 + 192xt^6 - 384xt^7, \cdots.$$

Thus, we have that

$$\begin{aligned} u(x,t) &= \lim_{n\to\leftrightarrow} u_n(x,t) \\ &= 3x(1 - 2t + +4t^2 - 8t^3 + 16t^4 - 32t^5 + 64t^6 - 128t^7 + \cdots) \\ &= \sum_{n=0}^{\infty}(-1)^n 2^n (3x)t^n \\ &= 3x/(1+2t). \end{aligned}$$

This is an exact solution.

Remark Starting with $u_0(x,t) = (a+1)x,\ u(x,t) = \lim_{n\to\infty} u_n(x,t) = (a+1)/(1+2t)\cdot$

As $a > 0$ (as special case: $a = 1$ that is example 1 [15], $a = 2,$ this is example 1 in this paper). Notice that $u_0(x,t) = ax + b$ or $u_0(x,t) = x^2 + x$ (similar polynomial case). More form.

$$\begin{aligned} u(x,t) &= \lim_{n\to\leftrightarrow} u_n(x,t) \\ &= (x^2 + x)(1 - 2t + +4t^2 - 8t^3 + 16t^4 - 32t^5 + 64t^6 + \cdots) \\ &= \sum_{n=0}^{\infty}(-1)^n 2^n (x^2 + x)t^n \\ &= (x^2 + x)/(1+2t). \end{aligned}$$

This is an exact solution.

Or $u_0(x,t) = x^k + x^{k-1} + x$ (similar polynomial case).more form.

$$u(x,t) = \lim_{n\to\leftrightarrow} u_n(x,t) = (x^k + x^{k+1} + x)/(1+2t).$$

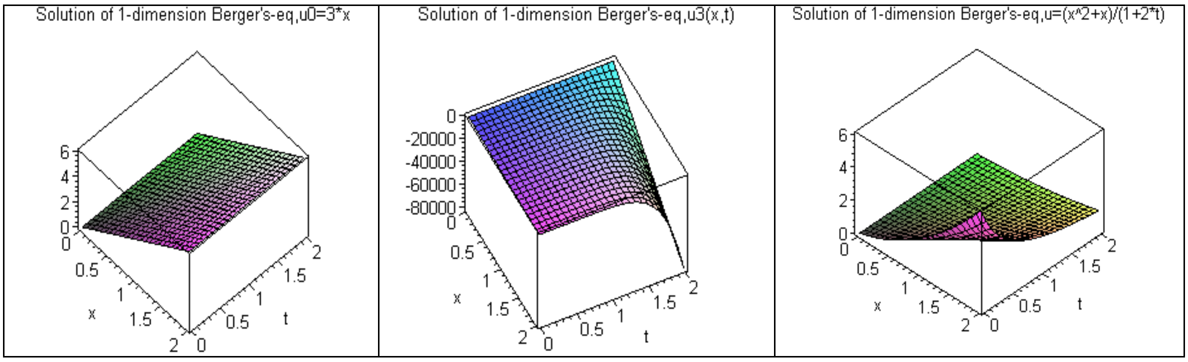

Fig 4

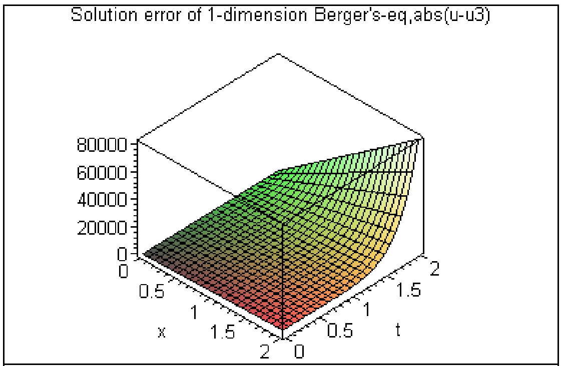

Fig 5

Example 6.2 Consider the following equation (similar as example 1 [15]):

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = 10x, t > 0. \end{cases} \tag{6.2}$$

For this example, by above Example 6.1, can be written as $a = 9$. Starting with $u_0(x,t) = 10x$. Thus, we have that an exact solution.

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = 10x/(1+2t).$$

VI-2 Two-dimensional Burger's equation

Similar as example 3 [15], we get following example 2 for two-dimension case.

Example 6.3 Consider the system of Burger's equation in the following equation (similar as example 3 [15]):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{R}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}),\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{R}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}). \tag{6.3}$$

with initial conditions: $u_0(x,y,0) = x + y, v_0(x,y,0) = x - y.$ We have by iterative [15]:

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t [\frac{\partial u_n}{\partial \tau} + u_n\frac{\partial u_n}{\partial x} + v_n\frac{\partial u_n}{\partial y} - \frac{1}{R}(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2}]d\tau,$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) - \int_0^t [\frac{\partial v_n}{\partial \tau} + u_n\frac{\partial v_n}{\partial x} + v_n\frac{\partial v_n}{\partial y} - \frac{1}{R}(\frac{\partial^2 v_n}{\partial x^2} + \frac{\partial^2 v_n}{\partial y^2}]d\tau. \tag{6.4}$$

Consider the initial approximations $u_0(x,y,0) = x + y, v_0(x,y,0) = x - y.$ and applying VIM formula (6.4), other term of the sequence are computed as follows:

$$u_1(x,y,t) = x + y - 2xt, v_1(x,y,t) = x - y - 2yt.$$

$$u_2(x,y,t) = x + y - 2xt + 2xt^2 + 2yt^2 - (4/3)xt^3,$$

$$v_2(x,y,t) = x - y. - 2yt. + 2xt^2 - 2yt^2 - (4/3)yt^3.$$

$$\begin{aligned} u(x,y,t) &= \lim_{n\to\leftrightarrow} u_n(x,y,t) \\ &= x(1 + 2t^2 + 4t^4 + \cdots) + y(1 + 2t^2 + \cdots) - 2xt(1 + 2t^2 + \cdots) \\ &= (x + y - 2xt)/(1 - 2t^2). \end{aligned}$$

$$\begin{aligned} v(x,y,t) &= \lim_{n\to\leftrightarrow} v_n(x,y,t) \\ &= x(1 + 2t^2 + 4t^4 + \cdots) - y(1 + 2t^2 + \cdots) - 2yt(1 + 2t^2 + \cdots) \\ &= (x + y - 2xt)/(1 - 2t^2), \end{aligned}$$

which are exact solutions $(|t| < \sqrt{1/2})$. We omit the detail stating for this results.

VII. Conclusion

In this Letter, we apply the variation iteration method to integral-differential equation ,and extend some results in [3]- [4]-[5]. The obtained solution shows the method is also a very convenient and effective for various integral-differential equations, only one iteration leads to exact solutions.

Acknowledgements

This work is supported by the Natural Science Foundation (No.11ZB192) of Sichuan Education Bureau and the key program of Science and Technology Foundation (No.11Zd1007) of Southwest University of Science and Technology.

REFERENCES

[1]. D. Guo, V. Lashmikantham, Nonlinear Problems in abstract Cones(Academic Press, Inc., Boston, New York, 1988).
[2]. S.Y. Xu, New fixed point theorems for 1-set-contractive operators in Banach spaces, Nonlinear Anal.67,2007,938-944.
[3]. N. Chen, and J.Q. Chen, New fixed point theorems for 1-set-contractive operators and variation iteration method, Jp Journal of fixed point theory and application,Vol.6(3),2011,147-162.
[4]. N. Chen, J.Q. Chen, Operation equation and application variation iteration method, Applied Mathematics,3 (8),2012,857-863.
[5]. N.Chen , J.Q.Chen, Some approximation in cone metric space and variation iterative method, Applied Mathematics, 3(12),2012,2007-2018.
[6]. C. X. Zhu, Generalization of Krasnoselskki's theorem and Petrshyn's theorem, Applied mathematics letters 19,2006, 628-632.
[7]. Asghar Ghorbani, Jafar Sabaeri–Nadjafi, An effective modification of He's variational iteration method, Nonlinear Analysis: Real World Application10,2009, 2828-2833.
[8]. Abdul–Majid Wazwaz, The variational iterative method :A reliable analytic tool for solving linear and nonlinear wave equations, Computers & mathematics with applications ,54,2007,926-932.
[9]. S.Q.Wang,Ji-Huan He, Variational iterative method for solving integro-differential equations, Physics Letters A 367,2007,188-191.
[10]. J. Biazr, H. Ghazvini, He's iterative method for forth-order parabolic equation,An international Journal of computers and mathematics with applications,54,(2007,1047-1054.
[11]. N. Chen, Baodan Tian and Jiqian Chen, A class of new fixed point theorems and for 1-set -contractive operators and some variation iterative method,[J]. Physics Procedia 33,2012,1932-1938.
[12]. Yousry S. Hanna, On Approximate solutions of second-order linear partial differential equations, Applied Mathematics,3,2012,1001-1007.
[13]. Elcin YusufogIu, The variation iteration method for studying the Klein-Gordon equation., Applied Mathematics Letters,21,2008,669-674.
[14]. Z.L.Yang, Nontrivial solutions of systems of nonlinear Hammerstein integral equations and applications, Acta Mathematica Scientia, 26(A2),2006,233-240.
[15]. A. Saandatmandi, and M. Dehghan, He's iterative method for solving a partial differential equation arising in modeling of water waves, Zeitschrift fur naturfoschung ,64,2009,783-787.