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On A Group with a Faithful Ordinary Representation

Chun P. B., Gau P. D., Babangida B. G., Tiyi P. A.

Abstract: *This communication provides an Improvement on an established result which has been proven earlier by Brauer in [1]. Braver proved that, for a solvable (soluble) group G with a faithful ordinary representation of degree m<P-1 for a Prime p, G has a normal, abelian sylow p–subgroup say Q.*

In this paper, the assumption that G is solvable is dropped and the existence of such an abelian, normal sylow-p subgroup is established with the provision that the degree of such a faithful ordinary representation is reduced from m<(p-1) to m< ½ (p-1). The prove of our claim is obtained by a series of lemmas using the method of contradiction.

Key Words: *A solvable group, sylow-P-group, faithful representation, irreducible representation, group character.*

I. INTRODUCTION

Brauer had early in 1943 [1] proved that for a solvable group G, which has a faithful ordinary representation of degree say "m" less than P-1, G has a normal abelian sylow p-subgroup. We observe that if G is a faithful KG-module V of dimension m≤P-1, then its sylow-P-subgroups are always abelian. Indeed, if Q is a sylow p-subgroup then the restriction irreducible constituents of V_Q divides $|Q|$. since m<P, this means that the irreducible constituents of V_Q all have dimension 1. Moreover, since V is faithful, this shows that Q is abelian. Obviously Q is also normal.

LEMMA 1.1

Let H be a normal subgroup of the group G. let χ be an irreducible ordinary character of G such that H⊄ Kerχ and suppose that some x€G satisfy $C_H(x) = 1$ then $\chi(x)=0$

Proof: We enumerate the irreducible ordinary characters of G so that $\chi_1, \chi_2, \ldots, \chi_r$ has H in their kernels and $\chi_{r+1}, \ldots, \chi_s$, does not contain H in their kernels. The condition that $C_H(x) = 1$ shows that $|C_G(x)| = |C_{G/H}^{(Hx)}|$. Indeed each Hy centralizing Hx in G/H contains exactly one element of $C_G(x)$. The characters $\chi_1, \ldots \chi_r$ correspond to a complete set G of irreducible characters for G/H. So the usual orthogonality relations give

$\sum_{i=1}^{r} \quad |\chi_i(x)|^2 = C_{G/H}(Hx)| = |C_G(x)| = \sum_{i=1}^{r} |\chi_i(x)^2|$

Hence $\chi_{r+1}^{(x)} = \ldots = \chi_s(x) = 0$.

MAIN RESULT

Our main result which does not assume that G is solvable is given in the theorem 1.2 below.

THEOREM 1.2:

Let G be a group with a faithful ordinary representation of degree m< ½ (P-1), then G has a normal abelian sylow P- subgroup say Q.

Proof: To prove the theorem, we shall suppose that the theorem is false and then obtain a contradiction. To achieve this, we now assume that G is a minimal counter example. Thus G has a faithful ordinary character χ of degree m< ½ (P-1) and clearly G is not abelian so m>1

Let Q be a sylow P-subgroup of G, it implies that Q is abelian. We now proceed on the proof in a series of lemmas;

LEMMA 1.3

For each normal subgroup H of G with H ≠ G, H ∩ Q ≤ Z (G)

Proof: Since G is a minimal counter example, $Q_P(H)$ is the sylow P-subgroup of H.

Put L = $C_G(O_P(H)$ Δ G. Since Q is abelian, Q ≤ L. if L = G, then Q =O_P(L) by minimalism of G and so Q Δ G since L Δ G. this is contrary to the hypothesis, so L = G, hence H ∩ Q≤ O_P(H) ≤ Z (G) and this implies that H ∩ Q≤ Z (G). ◊

LEMMA 1.4

Let G^1 be the derived group of G, then G = G^1

Proof: Since Q ≤ Z (G), it follows from Lemma 1.3 that QG^1 = G. Suppose that G^1 ≠ G, then G^1 ∩ Q = 1. Since Q is not normal in G, we can choose a prime q so that q divides $|G:N_G(Q)|$

Let T be a sylow q-subgroup of G for each x € G, $x^{-1}Tx$ is also a sylow q-subgroup of G^1 so $x^{-1}Tx = y^{-1}Ty$ for some y € G^1.

This shows that x€ $N_G(T) \leq N_G(T)G^1$, this holds for all x€G

So G = $N_G(T)G^1$. Hence $N_G(T)$ contains a sylow P-subgroup T_1, say of G. consider now the subgroup TT_1, then TΔTT_1, so TT_1 is solvable on the other hand, since T_1 is conjugate to Q, the choice of Q shows that T≤ $N_G(T_1)$. So T_1 is not normal in TT_1, which contradicts Brauer's theorem.

Thus G = G^1 as claimed.

LEMMA 1.5

χ is irreducible.

Proof: Suppose χ is not irreducible, then we can write $\chi = \chi_1+\chi_2$ as a sum of characters. By this choice χ , χ_1, χ_2 are not faithful and so by the choice of G, $O_Q(G/Ker\chi_i)$ is the sylow P-subgroup of $G/ker\chi_i$ (i=1,2,3…)

Then O_Q ($G/Ker\chi_1$ X$G/Ker\chi_2$) is the sylow P-subgroup of ($G/Ker\chi_1$ X$G/Ker\chi_2$) since $Ker\chi_1 \cap Ker\chi_1 = Ker\chi = 1$. There is an injection homeomorphism of G into this latter group given by $\chi \rightarrow (\chi Ker\chi_1, \chi Ker\chi_2)$. But that implies that $O_Q(G)$ is the sylow P-subgroup of G contrary to the hypothesis i.e G is minimal. Thus χ is irreducible.◊

LEMMA 1.6

Z (G) is the unique maximal normal subgroup of G. It is a P-group and G/Z (G) is simple and non – cyclic.

Proof: Let H be a maximal normal subgroup of G, then the subgroup QH ≠ G; for otherwise $G/H \equiv Q/Q\cap H$ is abelian and so H = G by lemma 1.4. But by Lemma 1.3, $Q\cap H \leq Z(G) = Z(G)\cap G^1$; so we have that $Q\cap H = 1$, since QH is a proper subgroup of G.

The minimalism of G shows that Q Δ QH. But H Δ QH, so QH = QXH. This shows that $Q \leq C_G (H) \Delta G$. Since $Q \not\subset Z(G)$, Lemma 1.3 now shows that $C_G(H) = G$, so $H \leq Z(G)$. Since G is non-abelian and H is maximal subgroup, H = Z(G). Since $H\cap Q = 1$, we have that Z(G) is a P-group. Since H is a maximal normal subgroup, G/Z(G) is simple. Finally G/Z(G) is non-cyclic since otherwise G would be abelian ◊

LEMMA 1.7

Let $N = N_G(Q)$ and No:= $\{x\varepsilon N/C_Q^{(x)} \neq 1\}$ then

(a) $x^{-1}Q x \cap Q = 1$ for all x ε G/N

(b) $x^{-1}No\, x \cap No = $ No if x εN
Z(G) if xεG/N

REMARK: Before we continue, we introduce the following notations. Let $|G| = P^e$ $|Z(G)| = P^e$ and $|N| = P^e h$.

Let $\chi_1 = I_{G}, \ldots \chi_2, \cdots, \chi_s$ be the irreducible ordinary characters of G and write

$(\chi_i)N = \alpha_i+\beta_i$, (i =1,2,3,…) where α_i is the sum of the remaining irreducible constituents of $(\chi_i)N$, none of which contain Q in their kernels and β_i is the sum of the remaining irreducible constituents. Since the representation affording χ_i is scalar on Z(G), then $|\beta_i(x)| = \beta_i(1)$ for all xεQZ(G).

LEMMA 1.8

For each i>1 we have that, $\beta_i (1)^2 p^e < \chi_i (1)^2$.

Proof: It follows from (b) of lemma 1.7 that No/Z(G) has $|G:N|$ distinct conjugates that are mutually disjoint. Therefore for i>1,

$\langle \chi_i , \chi_1 \rangle = \frac{1}{g} \sum_{x\varepsilon G} |\chi_1(x)|^2 \geq |\frac{G:N}{g}| \sum_{x\in No/Z(G)} |\chi_i(x)|^2 + \frac{1}{g} \chi_i (1)^2$.

For each irreducible character , χ_i, $|\chi_i(x) = \chi_i (1)$

whenever x ЄZ(G)

therefore $1> \frac{1}{|N|} \{\sum_{x\varepsilon N0} |\chi_i(x)|^2 - |Z(G)| = \chi_i (1)\}$

$\frac{1}{|N|} \sum_{x\varepsilon N0} \{ |\alpha_i(x)^2| + \alpha_i(x) \cancel{\beta_i(x)} + \cancel{\alpha_i(x)} \beta_i (x) + |\beta_i (x)^2| \} - \frac{1}{hp1/2^e} \chi_i (1)^2$

$\geq \langle \alpha_i , \alpha_i \rangle + 2 \langle \alpha_i , \beta_i \rangle + \frac{1}{h} \beta_i (1)^2 - \frac{1}{hp^e} \chi_i (1)^2$

Since $\alpha_{i,} = 0$ on N/No by lemma 1.1 and $|\beta_i (x)| = \beta_i (1)$

On QZ(G) by the definition of β_i . However, the definition of α_i and β_i shows that

$\langle \alpha_i , \beta_i \rangle$. Moreover since $\chi_i \neq \chi_i = I_G$, lemma 1.6 shows that $\underline{Q\leq}$ $ker\chi_i$. So $\alpha_I \neq 0$.

Therefore we conclude that;

$\chi_i(1)^2 = P^e \{\beta_i(1)^2 + h\langle \alpha_i , \alpha_i \rangle - h\}$

$\geq P^e \{\beta_i (1)^2$

LEMMA 1.9 $(I_Q)^G = \sum_{i=1}^{s} \beta_i(1)^2 \chi_i$

Proof: By Frobenius reciprocity,

$\langle I^G_{Q,} \chi_i \rangle = \langle I_Q, (\chi_i)_Q \rangle = \langle I_{Q,} ,(\alpha_i)_Q+(\beta_i)_Q \rangle$.

By the definition of α_i and $\beta_{i,}$

we have that; $\langle I_Q, (\alpha_i)_Q = 0 \rangle$ an $\langle I_Q, (\beta_i)_Q \rangle = \beta_i(1)$ and soothe result follows.

PROOF OF MAIN RESULT (THEOREM 1.2)

We first show that m ≤ 1+ $\sum_{i=2}^{s} \langle \chi_i, \chi\bar{\chi} \rangle (\beta_i)(1)$. Now by lemma 1.9 we have that;

$\sum_{i=2}^{s} \langle \chi_i, \chi\bar{\chi} \rangle (\beta_i)(1) = \langle (I_Q)^G, \chi\bar{\chi} \rangle$

$= \langle I_Q, (\chi\bar{\chi})_Q \rangle$

$= \langle \chi_Q, \chi_Q \rangle = \chi(1) = m$

Since χ_Q is a sum of χ(1) characters of degree 1. Since $\langle \chi_i, \chi\bar{\chi} \rangle \beta_i(1) = \langle \chi, \chi \rangle = 1$. The assertion follows. Therefore from lemma 1.8, it follows that;

$m \le 1 + \frac{1}{p1/2^s} \sum_{i=2}^{s} \langle \chi_i, \chi\bar{\chi} \rangle (\chi_i)(1)$

$= 1 + \frac{1}{p1/2^s} \{\langle I^G, \chi\chi \rangle - 1\}$

Because $I^G = \sum_{i=2}^{s} \chi_i(1)\chi_i$

Since $\langle I^G, \chi\bar{\chi} \rangle = \chi(1)^2 = m^2$. This shows that $p^{1/2s} \le \frac{m^2-1}{m-1} = m+1$.

But $m + 1 < \frac{1}{2}(p+1)$ by hypothesis and so $m = 1$.

However by Lemma 1.4 this means that the hypothesis of Brauer's theorem is satisfied, and that shows our counter example is impossible. This completes the proof of our main results (Theorem 1.2).

II. CONCLUSION

We have been able to establish through a series of Lemmas that if the condition on Bauer's theorem that the group G be soluble is dropped, then the condition that G be of degree say $m < (p-1)$ for a prime P, would now be that G is of degree $m < \frac{1}{2}(p-1)$ and the result would still hold i.e there would exist a normal abelian sylow P-subgroup say Q for G provided G has an ordinary faithful representation.

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