The Method of Repeated Application of a Quadrature Formula of Trapezoids and Rectangles to Determine the Values of Multiple Integrals

Nazirov Sh. A.

Tashkent University of Information Technologies, Tashkent, Uzbekistan

Abstract: *The work deals with the construction of multi-dimensional quadrature formulas based on the method of repeated application of the quadrature formulas of rectangles and trapezoids, to calculate the multiple integrals' values. We've proved the correctness of the quadrature formulas.*

Keywords: *multi-dimensional integrals, cubature formulas, re-use method.*

I. INTRODUCNION

The paper is devoted to developing the multiple cubature formulas for the values of n-fold integrals calculation by means of methods of repeated application of, quadrature formulas of trapezoid and rectangles.

According to mathematical analysis multiple integrals can be calculated by recalculating single integrals. The essence of this approach is the following. Let the domain of integration is limited to single-valued continuous curves [1]

$y=\varphi(x),\;\; y=\psi(x)\;\;\; (\varphi(x)\le\varphi(x))$

and two vertical lines $x=a,\;\; x=b$ (Fig.1).

Placing the known rules in the double integral

$$I=\iint\limits_{(D)} f(x,y)dxdy \qquad (1)$$

limits of integration, we obtain

$$\iint\limits_{(D)} f(x,y)dxdy=\int\limits_a^b dx\int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy.$$

Let

$$F(x)=\int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy. \qquad (2)$$

Then

$$\iint\limits_{(D)} f(x,y)dxdy=\int\limits_a^b F(x)dx. \qquad (3)$$

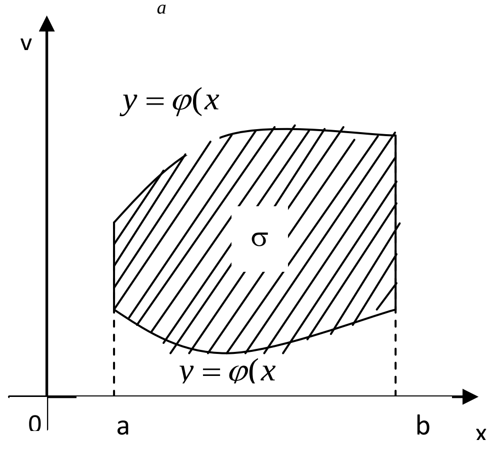

Fig.1

Applying to a single integral in the right-hand side of (3), one of the quadrature formulas, we have

$$\iint\limits_{(\sigma)} f(x,y)dxdy=\sum_{i=1}^{n} A_i F(x_i), \qquad (4)$$

where - $x_i\in[a,b]\;\;(i=1,2,...,n)$ and A_i are some constant coefficients. In turn, the value

$$F(x_i) = \int_{\varphi(x_i)}^{\psi(x_i)} f(x_i, y)dy$$

can also be found on some quadrature formulas

$$F(x_i) = \sum_{j=1}^{m_i} B_{ij} f(x_i, y_j),$$

where B_{ij} iy - appropriate constants.

From (4) we obtain

$$\iint_{(D)} f(x, y)dxdy = \sum_{i=1}^{n} \sum_{j=1}^{m_i} A_i B_{ij} f(x_i, y_j), \quad (5)$$

where A_i and B_{ij} iy - are known constants.

Geometrically, this method is equivalent to the calculation of the volume I, given by the integral (2) by means of cross-sections.

For cubature formulas of the type (3), the general comments, relating to the computation of simple integrals, are valid with appropriate modifications.

In three dimensions (3) has the form

$$\iiint_{(D)} f(x, y, z)dxdydz = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{k=1}^{t_i} A_i B_{ij} C_{ij,k} f(x_i, y_j, z_k).$$

Now these results are generalized to compute the values of multiple integrals

$$\iint \underset{(D)}{...} \int f(x_1, x_2, x_k)dx_1 dx_2 ...dx_k =$$

$$= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_n=1}^{n_k} A_{i_1}^{(1)} A_{i_1 i_2 i_3}^{(2)} A_{i_1 i_2 i_3}^{(3)} ... A_{i_1 i_2 i_k}^{(k)} f(x_{1_i}, x_{2_i}, ..., x_{k_i}),$$

Where $A_{i_1}^{(1)}, A_{i_1 i_2}^{(2)}, ..., A_{i_1 i_2 ... i_k}^{(k)}$ - are known constants.

This approach can be applied to calculate the approximate value of multiple integrals using appropriate quadrature formulas. In this case, we apply the quadrature formula of trapezoid.

II. ALGORITHM DISCRIPTION

Theorem 1. Let the function

$$y = f(x_1, x_2, ..., x_n)$$

Is continuous in a bounded domain D. Then the formula for the approximate values of the n-fold integral calculating

$$J_n = \iint \underset{(D)}{...} \int f(x_1, ..., x_n)dx_1 dx_2, ..., dx_n, \quad (6)$$ based on the trapezoidal rule has the form

$$J_n = \prod_{i=1} h_i \sum_{p_1=0}^{n_1-1} \sum_{p_2=0}^{n_2-} \sum_{p_3=0}^{n_3-1} ... \sum_{p_n=0}^{n_n-1} \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} ... \sum_{i_n=0}^{1} f_{p_1+i_1, p_2+i_2, ..., p_n+i_n}$$

Here D is n-dimensional domain of integration of the form

$$a_i \le x_i \le A_i, \quad i = \overline{1, n}; \quad (7)$$

$$h_i = \frac{A_i - a_i}{2}$$

and it is believed that each interval (5), is respectively, divided into $n_1, n_2, ..., n_n$ parts.

Proof. Approximate formula for calculating the values of the one-dimensional integral in this case is

$$J_1 = \int_{x_0}^{x_1} f(x)dx = \frac{h_x}{2}[f(x_0) + f(x_1)] = \frac{h_x}{2} \sum_{i=0}^{1} f_i, \quad (8)$$

where, $h_x = \frac{b-a}{n}$, n - the number of the interval divisions into n parts. In (4) n = 1 .

Now we define the approximate calculation formula for double integrals

$$J_2 = \iint_{(D)} f(x, y)dxdy \quad (9)$$

where D is the two-dimensional area of integration.
It is generally believed that the region of integration in this case is within the rectangle, that is,
$a \le x \le b,\ c \le y \le d$.
Then we rewrite the integral (7) in the form

$$J_2 = \int_a^b dx \int_c^d f(x,y)dy. \qquad (10)$$

Using the trapezoidal rule to calculate the inner integral, we obtain

$$J_2 = \frac{h_y}{2}\left[\int_a^b f(x,y)dx + \int_a^b f(x,y)dx\right]. \qquad (11)$$

Now to each integral we use the trapezoid formulas and the result is

$$J_2 = \frac{h_y}{2}\left\{\frac{h_x}{2}\left[f(x_0,y_0)+f(x_1,y_0)\right]+\frac{h_x}{2}\left[f(x_0,y_1)+f(x_1,y_1)\right]\right\} = \\ \frac{h_x h_y}{4}\left[f(x_0,y_0)+f(x_1,y_0)+f(x_0,y_1)+f(x_1,y_1)\right] = \frac{h_x h_y}{4}\sum_{i=0}^{1}\sum_{j=0}^{1} f(x_i,y_j), \qquad (12)$$

where $h_y = \dfrac{d-c}{m}$.

Similarly, for the approximate calculation of the values of the triple integral we obtain the following formula:

$$J_3 = \int_a^A\int_b^B\int_c^C f(x,y,z)dxdydz = \frac{h_x h_y h_z}{8}\left[f(x_0,y_0,z_0)+f(x_0,y_0,z_1)+f(x_0,y_1,z_0)+\right. \\ \left.+f(x_0,y_1,z_1)+f(x_1,y_0,z_0)+f(x_1,y_0,z_1)+f(x_1,y_1z_0)+f(x_1,y_1z_0)+f(x_1,y_1,z_1)\right] = \\ \frac{h_x h_y h_z}{8}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} f(x_i,y_j,z_k), \qquad (13)$$

where $h_z = \dfrac{C-c}{p}$.

The four-dimensional integral is defined with 16 members:

$$J_4 = \int_a^A\int_b^B\int_c^C\int_d^D f(x,y,z,u)dxdydzdu = \frac{h_x h_y h_z h_u}{16}\left[f(x_0,y_0,z_0,u_0)+f(x_0,y_0,z_0,u_1)+f(x_0,y_0,z_1,u_0)+\right. \\ +f(x_0,y_0,z_1,u_1)+f(x_0,y_1,z_0,u_0)+f(x_0,y_1,z_0,u_1)+f(x_0,y_1,z_1,u_0)+f(x_0,y_1,z_1,u_1)+ \\ +f(x_1,y_0,z_0,u_0)+f(x_1,y_0,z_0,u_1)+f(x_1,y_0,z_1,u_0)+f(x_1,y_0,z_1,u_1)+f(x_1,y_1,z_0,u_0)+ \\ \left.+f(x_1,y_1,z_0,u_1)+f(x_1,y_1,z_1,u_1)\right] = \frac{h_x h_y h_z h_u}{16}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1} f(x_i,y_j,z_k,u_l); \qquad (14)$$

Here $h_u = \dfrac{D-d}{t}$.

Considering (8.10), we present formulas for the approximate calculation of the n-fold integral on the trapezoidal:

$$J_n = \int_{a_1}^{A_1}\int_{a_2}^{A_2}\int_{a_3}^{A_3}\ldots\int_{a_n}^{A_n} f(x_1,x_2,\ldots,x_n)dx_1dx_2dx_3\ldots dx_n = \frac{1}{2^n}\prod_{i=1}^{n} h_i \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1}\ldots\sum_{i_n=0}^{1} f(x_{1i_2},x_{2i_2},\ldots,x_{ni_n}) \qquad (15)$$

Then for the formulas (6) and (8) - (11) we derive the general trapezoid formula. For this purpose, one-dimensional integral for integral intervals [a, b] is divided by n equal parts: $[x_0,x_1],\ldots,[x_{n-1},x_n],\ x_i = x_0 + ih_x$.

The two-dimensional integral of the $[a,b]\times[c,d]$ region of integration is divided, respectively, by n and m parts:
$[x_0,x_1],\ldots,[x_{n-1},x_n];\quad [y_0,y_1],\ldots,[y_{m-1},y_m]$

$$x_i = x_0 + ih_x, \\ y_j = y_0 + jh_y. \qquad (16)$$

In the same way ,the three-, four-, etc. n-dimensional integration region is divided by the corresponding parts.
In this approach, the formula (6) takes the form

$$J_1 = \int_a^b f(x)dx = \frac{h}{2}\sum_{i=0}^{n-1}(y_i + y_{i+1}) = \frac{h}{2}\sum_{i=0}^{n-1}(f_i + f_{i+1}) = \frac{h}{2}\sum_{i=1}^{n}(f_{i-1} + f_i) = \tag{17}$$
$$= \frac{f_0 + f_n}{2} + \sum_{i=0}^{n-1} f_i .$$

The general trapezoid formula for double integrals calculating has the form

$$J_2 = \frac{h_x h_y}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f_{i,j} + f_{i,j+1} + f_{i+1,j} + f_{i+1,j+1}\right] = \tag{18}$$
$$= \frac{h_x h_y}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{i_1=0}^{1}\sum_{i_2=0}^{1} f_{i+i_1 j+i_2} .$$

The following is a general formula for the trapezoidal three-and four-fold integrals calculating

$$J_3 = \int_a^A\int_b^B\int_c^C f(x,y,z)dxdydz = \frac{h_x h_y h_z}{8}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1}\left[f_{i,j,k} + f_{i,j,k+1} + f_{i,j+1,k} + \right. \tag{19}$$
$$\left. + f_{i,j+1,k+1} + f_{i+1,j,k} + f_{i+1,j,k+1} + f_{i+1,j+1,k} + f_{i+1,j+1,k} + f_{i+1,j+1,k+1}\right] =$$
$$= \frac{h_x h_y h_z}{8}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1}\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1} f_{i+i_1, j+i_2, k+i_3} ,$$

$$J_4 = \int_a^A\int_b^B\int_c^C\int_d^D f(x,y,z,u)dxdydzdu = \frac{h_x h_y h_z h_u}{16}\left[f_{i,j,k,l} + f_{i,j,k,l+1} + f_{i,j,k+1,l} + \right.$$
$$+ f_{i,j,k+1,l+1} + f_{i,j+1,k,l} + f_{i,j+1,k,l+1} + f_{i,j+1,k+1,l} + f_{i,j+1,k+1,l+1} + f_{i+1,j,k,l} + f_{i+1,j,k,l} +$$
$$\left. + f_{i+1,j,k+1,l} + f_{i+1,j,k+1,l+1} + f_{i+1,j+1,k,l} + f_{i+1,j+1,k,l+1} + f_{i+1,j+1,k+1,l} + + f_{i+1,j+1,k+1,l+1}\right] = \tag{20}$$
$$= \frac{h_x h_y h_z h_u}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1}\sum_{l=0}^{t-1}\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1}\sum_{i_4=0}^{1} f_{i+i_1, j+i_2, k+i_3, l+i_4} .$$

Based on the above formula, the approximate calculation of n-fold integral, using the formula (D), takes the form

$$J_n = \frac{1}{2^n}\prod_{i=1}^{n}(A_i - a_i)\sum_{p_1=0}^{n_1-1}\sum_{p_2=0}^{n_2-1}\sum_{p_3=0}^{n_3-1}\cdots\sum_{p_n=0}^{n_n-1}\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{1} f_{p_1+i_1, p_2+i_2, \dots, p_n+i_n} . \tag{21}$$

Thus the theorem is completely proved.

Theorem 2. Approximation formulas for determining the values of the double integral, obtained by the repeated use of the trapezoid quadrature formula is calculated as follows:

$$\iint_{(D)} f(x,y)dxdy = \frac{h_x h_y}{4}\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_{ij} f_{ij} , \tag{22}$$

where

$$\lambda_{ij} = \begin{bmatrix} 1_* & 2 & 4 & \dots & 2 & 1 \\ 2 & 4 & 4 & \dots & 4 & 2 \\ 2 & 4 & 4 & \dots & 4 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 4 & 4 & \dots & 4 & 2 \\ 1 & 2 & 2 & \dots & 2 & 1 \end{bmatrix} . \tag{23}$$

Proof. Let's use the formula (12). Revealing this amount and similar terms, we obtain the following formula:

$$\frac{h_x h_y}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f_{ij} + f_{ij+1} + f_{i+1,j} + f_{i+1,j+1}\right] =$$
$$= \frac{h_x h_y}{4}\left[f_{11} + 2f_{12} + 2f_{13} + \dots + 2f_{1n-1} + f_{1,n}\right.$$
$$2f_{21} + 4f_{22} + 4f_{23} + \dots + 4f_{2,n-1} + 4f_{2,n} \tag{24}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$4f_{n-1,1} + 4f_{n-1,2} + 4f_{n-1,3} + \dots + 4f_{n-1,n-1} + 4f_{n-1,n} .$$
$$\left. f_{n,1} + 2f_{n,2} + f_{n,3} + \dots + 2f_{n,n-1} + f_{n,n}\right]$$

The (17) can form the matrix elements - λ_{ij} in the form (16) and the result (15), indicating the proof of the theorem.

Theorem 3. Approximate formula for calculating the value of triple integrals, obtained by the repeated use of the trapezoid quadrature formula, is calculated as follows:

$$\iiint\limits_{(D)} f(x,y,z)dxdydz = \frac{h_x h_y h_z}{8}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{S}\lambda_{ijk} f_{ijk}\ , \tag{25}$$

where

$$\lambda_{ijk} = \begin{pmatrix} 1 & 2 & 2 & 2 & \dots & 2 & 1 \\ 2 & 4 & 4 & 4 & \dots & 4 & 2 \\ 2 & 4 & 4 & 4 & \dots & 4 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 4 & 4 & 4 & \dots & 4 & 2 \\ 1 & 2 & 2 & 2 & \dots & 2 & 1 \end{pmatrix}, \qquad i = 1; n,$$

$$\lambda_{ijk} = \begin{pmatrix} 2 & 4 & 4 & \dots & 4 & 4 & 2 \\ 4 & 8 & 8 & \dots & 8 & 4 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 4 & 8 & 8 & \dots & 8 & 4 & 2 \\ 2 & 4 & 4 & \dots & 4 & 4 & 2 \end{pmatrix}, \qquad i = \overline{2, n-1}. \tag{26}$$

Proof. Let's use (13). Revealing this amount and similar terms, we obtain

$$\begin{gathered}
f_{1,1,1} + 2f_{1,1,2} + 2f_{1,1,3} + 2f_{1,1,4} + \dots + 2f_{1,1,n-1} + f_{1,1,n} \\
2f_{1,2,1} + 4f_{1,2,2} + 4f_{1,2,3} + 4f_{1,2,4} + \dots + 4f_{1,1,n-1} + 2f_{1,2,n} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
2f_{1,n-1,1} + 4f_{1,n-1,2} + 4f_{1,n-1,3} + 4f_{1,n-1,4} + \dots + 4f_{1,n-1,n-1} + 2f_{1,n-1,n} \\
f_{1,n,1} + 2f_{1,n,2} + 2f_{1,n,3} + 2f_{1,n,4} + \dots + 2f_{1,n,n-1} + f_{1,n,n} \\
2f_{2,1,1} + 4f_{2,1,2} + 4f_{2,1,3} + 4f_{2,1,4} + \dots + 4f_{2,1,n-1} + 2f_{2,1,n} \\
4f_{2,2,1} + 8f_{2,2,2} + 8f_{2,2,3} + 8f_{2,2,4} + \dots + 8f_{2,2,n-1} + 4f_{2,2,n} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
4f_{2,10^{-1},1} + 8f_{2,10^{n-1},2} + 8f_{2,10,3} + 8f_{2,10^{n-1},4} + \dots + 8f_{2,10^{n-1},n-1} + 4f_{2,10^{n-1},n} \\
2f_{2,11,1} + 4f_{2,11,2} + 4f_{2,11,3} + 4f_{2,11,4} + \dots + 4f_{2,11,n-1} + 2f_{2,11,n} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
2f_{n-1,1,1} + 4f_{n-1,1,2} + 4f_{n-1,1,3} + 4f_{n-1,1,4} + \dots + 4f_{n-1,1,n-1} + 2f_{n-1,1,n} \\
4f_{n-1,2,1} + 8f_{n-1,2,2} + 8f_{n-1,2,3} + 8f_{n-1,2,4} + \dots + 8f_{n-1,2,n-1} + 4f_{n-1,2,n} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
4f_{n-1,n-1,1} + 8f_{n-1,n-1,2} + 8f_{n-1,n-1,3} + 8f_{n-1,n-1,4} + \dots + 8f_{n-1,n-1,n-1} + 4f_{n-1,n-1,n} \\
2f_{nn,n,1} + 4f_{n-1,n,2} + 4f_{n-1,n,n} + 4f_{n-1,n,n} + \dots + 4f_{n-1,n,n-1} + 2f_{n-1,n,n} \\
f_{n,1,1} + 2f_{n,1,2} + 2f_{n,1,3} + 2f_{n,1,4} + \dots + 2f_{n,1,n-1} + f_{n,1,n} \qquad (27) \\
2f_{n,2,1} + 4f_{n,2,2} + 4f_{n,2,3} + 4f_{n,2,4} + \dots + 4f_{n,2,n-1} + 2f_{n,2,n} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
2f_{n,n-1,1} + 4f_{n,n-1,2} + 4f_{n,n-1,3} + 4f_{n,n-1,4} + \dots + 4f_{n,n-1,n-1} + \dots + 2f_{n,n-1,n} \\
f_{n,n,1} + 2f_{n,n,2} + 2f_{n,n,3} + 2f_{n,n,4} + \dots + 4f_{n,n,n-1} + f_{n,n,n}.
\end{gathered}$$

From (19) we form the λ_{ijk} elements, resulting in the relation (18), which shows the proof of the theorem.

Theorem 4. Approximation formulas for the values determining of the quadruple integrals, obtained by the repeated use of the trapezoid quadrature formula, is calculated as follows:

$$\iiint\limits_{(D)}\int f(x,y,z,u)dxdydzdu = \frac{h_x h_y h_z h_u}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{l-1}\sum_{l=0}^{s-1}\lambda_{ijkl} f_{ijkl}\ , \tag{28}$$

$$\Lambda\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1,n-1} & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2,n-1} & A_{2n} \\ . & . & \dots & . & . \\ A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,n-1} & A_{n-1,n} \\ A_{n,1} & A_{n,2} & \dots & A_{n,n-1} & A_{nn} \end{bmatrix},$$

$$A_{n1} = A_{nn} = A_{1n} = \begin{bmatrix} 1 & 2 & 2 & 2 & \dots & 2 & 1 \\ 2 & 4 & 4 & 4 & \dots & 4 & 2 \\ . & . & . & . & \dots & . & . \\ 2 & 4 & 4 & 4 & \dots & 4 & 2 \\ 1 & 2 & 2 & 2 & \dots & 2 & 1 \end{bmatrix},$$

$$A_{n,j} = A_{1,j}\begin{bmatrix} 2 & 4 & 4 & \dots & 4 & 2 \\ 4 & 8 & 8 & \dots & 8 & 4 \\ . & . & . & . & \dots & . \\ 4 & 8 & 8 & \dots & 8 & 4 \\ 2 & 4 & 4 & \dots & 4 & 1 \end{bmatrix}, \quad j = \overline{1, n}-1,$$

$$A_{i1} = A_{in} = \begin{bmatrix} 2 & 4 & \dots & 4 & 2 \\ 4 & 8 & \dots & 8 & 4 \\ . & . & \dots & . & . \\ 4 & 8 & \dots & 8 & 4 \\ 2 & 4 & \dots & 4 & 2 \end{bmatrix}, \quad i = \overline{2, n-1}, \quad A_{ij} = \begin{bmatrix} 4 & 8 & \dots & 8 & 4 \\ 8 & 16 & \dots & 16 & 8 \\ . & . & \dots & . & . \\ 8 & 16 & \dots & 16 & 8 \\ 4 & 8 & \dots & 8 & 4 \end{bmatrix}, \quad \begin{matrix} i = 2,3,\dots,n-1, \\ j = 2,3,\dots,n-1 \end{matrix}$$

Proof. Disclose the amount (14) and obtain the following relations:

$$f_{1111} + 2f_{1112} + 2f_{1113} + 2f_{1114} + \dots + 2f_{111n-1} + f_{111,n}$$
$$2f_{1121} + 4f_{1122} + 4f_{1123} + 4f_{1124} + \dots + 4f_{112,n-1} + 2f_{112,n}$$
$$2f_{1131} + 4f_{1132} + 4f_{1133} + 4f_{1134} + \dots + 4f_{113,n-1} + 2f_{113,n}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$2f_{11,n-1,1} + 4f_{11,n-1,2} + 4f_{11,n-1,3} + 4f_{11,n-1,4} + \dots + 4f_{11,n-1,n-1} + 2f_{11,n-1,n}$$
$$f_{11n1} + 2f_{11n2} + 2f_{11n3} + 2f_{11n4} + \dots + 2f_{11n,n-1} + f_{111,n}$$
$$2f_{1211} + 4f_{1212} + 4f_{1213} + 4f_{1214} + \dots + 4f_{121,n-1} + 2f_{121,n}$$
$$4f_{1221} + 8f_{1222} + 8f_{1223} + 8f_{1224} + \dots + 8f_{122,n-1} + 4f_{122,n}$$
$$4f_{1231} + 8f_{1232} + 8f_{1233} + 8f_{1234} + \dots + 8f_{123,n-1} + 4f_{123,n}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$4f_{12n-1,1} + 8f_{12n-1,2} + 8f_{12n-1,3} + 8f_{12n-1,4} + \dots + 8f_{12n-1,n-1} + 4f_{12n-1,n}$$
$$2f_{12n1} + 4f_{12n2} + 4f_{12n3} + 4f_{12n4} + \dots + 4f_{12n,n-1} + 2f_{12n,n}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f_{1n11} + 2f_{1n12} + \dots + 2f_{1n-1,n-1} + f_{1n,n}$$
$$2f_{1n21} + 4f_{1n22} + \dots + 4f_{1n-1,2n-1} + 2f_{1n,2n}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$2f_{1n,n-1,1} + 4f_{1nn-1,2} + \dots + 4f_{1n,n-1,n-1} + 2f_{1nn-1,n}$$
$$f_{1nn1} + 2f_{1nn2} + \dots + 2f_{1nnn-1} + f_{1nn,n}$$

$$f_{2111} + 4f_{2112} + 4f_{2113} + \dots + 4f_{211,n-1} + 2f_{211,n}$$
$$4f_{2121} + 8f_{2122} + 8f_{2123} + \dots + 8f_{212,n-1} + 4f_{212,n}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$4f_{21n-1,1} + 8f_{21n-1,2} + 8f_{21n-1,3} + \dots + 8f_{21n-1,n-1} + 4f_{21n-1,n}$$
$$2f_{21n1} + 4f_{21n2} + 4f_{21n3} + \dots + 4f_{21n,n-1} + 2f_{21n,n}$$

$4f_{2211}+8f_{2212}+8f_{2213}+\ldots+8f_{221,n-1}+4f_{221,n}$
$8f_{2221}+16f_{2222}+16f_{2223}+\ldots+16f_{222,n-1}+8f_{222,n}$
$8f_{2231}+16f_{2232}+16f_{2233}+\ldots+16f_{223,n-1}+8f_{223,n}$
....
$8f_{22n-1,1}+16f_{22n-1,2}+16f_{22n-1,3}+\ldots+16f_{22n-1,n-1}+8f_{22n-1,n}$
$4f_{22n1}+8f_{22n2}+8f_{22n3}+\ldots+8f_{22n,n-1}+4f_{22n,n}$
....

$4f_{n-1,n11}+8f_{n-1,n12}+8f_{n-1,n13}+\ldots+8f_{n-1,n,1,n-1}+4f_{n-1,n,1,n}$
$8f_{n-1,n21}+16f_{n-1,n22}+16f_{n-1,n23}+\ldots+16f_{n-1,n,2,n-1}+8f_{n-1,n,2,n}$
....

$2f_{n,n-1,11}+4f_{n,n-1,12}+4f_{n,n-1,13}+\ldots+4f_{n,n-1,1,n-1}+2f_{n,n-1,1,n}$
$4f_{n,n-1,21}+8f_{n,n-1,22}+8f_{n,n-1,23}+\ldots+8f_{n,n-1,2,n-1}+4f_{n,n-1,2,n}$
....
$4f_{n,n-1,n-1,1}+8f_{n,n-1,n-1,2}+8f_{n,n-1,n-1,3}+\ldots+8f_{n,n-1,n-1,n-1}+4f_{n,n-1,n-1,n}$
$2f_{n,n-1,n,1}+4f_{n,n-1,n,2}+4f_{n,n-1,n,3}+\ldots+4f_{n,n-1,n,n-1}+2f_{n,n-1,n,n}$

$8f_{n-1,n,n-1,1}+16f_{n-1,n,n-1,2}+16f_{n-1,n,n-1,3}+\ldots+16f_{n-1,n,n-1,n-1}+8f_{n-1,n,n-1,n}$
$4f_{n-1,n,n,1}+8f_{n-1,n,n,2}+8f_{n-1,n,n,3}+\ldots+8f_{n-1,n,n,n-1}+8f_{n-1,n,n,n}$

$f_{nn11}+2f_{nn12}+2f_{nn13}+\ldots+2f_{nn1,n-1}+f_{nn1n}$
$2f_{nn21}+4f_{nn22}+4f_{nn23}+\ldots+4f_{nn2,n-1}+4f_{nn2,n}$
....
$2f_{nnn-1,1}+4f_{nnn-1,2}+4f_{nnn-1,3}+\ldots+4f_{nn,n-1,n-1}+4f_{nn,n-1,n}$
$f_{nnn1}+2f_{nnn2}+2f_{nnn3}+\ldots+2f_{nnnn-1}+f_{nnn,n}$

From these relations the formula (20) is formed, which fully demonstrates the proof of the theorem.

Now, we calculate the values of integrals by repeated application of quadrature formulas, based on the formula of rectangles.

The rectangle formula to calculate the values of the integrals is quite simple. In fact, here ξ_i is taken as $[x_{i-1}, x_i]$ midpoints. For a uniform grid $(h_i = h, \quad i=\overline{1,n})$ we obtained formulas of the form

$$J_1 = \int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{i=1}^{n} f_{i-1/2}, \qquad (29)$$

where $f_{i-1/2} = f(x_i - \frac{h}{2}), \quad x_0 = a, \quad x_n = b$.

Double integral in this case is calculated by the formula

$$J_2 \equiv \int_a^A\int_b^B f(x,y)dxdy = \int_a^A dx\int_b^B f(x,y)dy =$$
$$= \int_a^A\left[\frac{B-b}{n_2}\sum_{j=1}^{n_2} f\left(x, y_{j-1/2}\right)\right]dx = \frac{A-a}{n_1}\frac{B-b}{n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} f\left(x_{i-1/2}, y_{j-1/2}\right) = \qquad (30)$$
$$= \frac{A-a}{n_1}\frac{B-b}{n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} f_{i-1/2,j-1/2},$$

where $f_{i-1/2,j-1/2} = f\left(x_i - \frac{h_x}{n} y_j - \frac{h_y}{2}\right), \qquad n_1 = \frac{A-a}{h_x}, \quad n_2 = \frac{B-b}{h_y};$

h_x - Step on the OX axis, h_y y - a step on the OY axis.

In the same way we define cubature formulas for calculating the values of the three-and four-fold integrals

$$J_3 = \int_a^A\int_b^B\int_c^C f(x,y,z)dxdydz \approx \frac{A-a}{n_1}\frac{B-b}{n_2}\frac{C-c}{n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} f_{i-1/2,j-1/2,k-1/2}, \quad (31)$$

Where $f_{i-1/2,j-1/2,k-1/2} = f\left(x_i - \frac{h_x}{2}, y_j - \frac{h_y}{2}, z_k - \frac{h_z}{2}\right),$

$$J_4 \equiv \int_a^A\int_b^B\int_c^C\int_d^D f(x,y,z,u)dxdydzdu \approx \frac{A-a}{n_1}\frac{B-b}{n_2}\frac{C-c}{n_3}\frac{D-d}{n_4}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\sum_{l=1}^{n_4} f_{i-1/2,j-1/2,k-1/2,l-1/2}; \quad (32)$$

Here $f_{i-1/2,j-1/2,k-1/2,l-1/2} = f\left(x_i - \frac{h_x}{2}, y_j - \frac{h_y}{2}, z_k - \frac{h_z}{2}, u_l - \frac{h_u}{2}\right).$

On the basis of approximate formulas for calculating the values of one-, two-, three-and four-fold integrals, respectively, expressed by (21) - (24), the formula for calculating the values of n-fold integrals is given .

Theorem 5. Let the $y = f(x_1, x_2, ..., x_n)$ function is defined and continuous in a n-dimensional bounded Ω_n integration domain. Then the cubature formula, obtained by repeated application of the rectangles' formula, has the form

$$\int\int\int\underset{D}{\cdots}\int f(x_1,x_2,...,x_n)dx_1dx_2...dx_n = \prod_{i=1}^{n}\frac{A_i-a_i}{n_i}\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_n=1}^{n_n} f_{i_1-1/2,i_2-1/2,...,i_n-1/2}, \quad (33)$$

where $x_i \in [a_i, A_i], \quad i = \overline{1,n},$

$$f_{i_1-1/2,i_2-1/2,...,i_n-1/2} = f\left(x_{i_1} - \frac{h_1}{2}, x_{i_2} - \frac{h_2}{2}, ..., x_{i_n} - \frac{h_n}{2}\right).$$

Proof. The proof is given by induction.

From (21) - (24) we see that (25) holds for $n = 1;2;3;4$. We assume that (25) is valid for $n = k$:

$$\int\int\underset{D}{\cdots}\int f(x_1,x_2,...,x_k)dx_1dx_2...dx_k = \prod_{i=1}^{k}\frac{A_i-a_i}{k_i}\sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_k=1}^{k_k} f_{i_1-1/2,i_2-1/2,...,i_k-1/2}.$$

Now we show fairness at $n = k+1$:

$$\int\int\underset{(D)}{\cdots}\int f(x_1,x_2,...,x_k,x_{k+1})dx_1dx_2...dx_kdx_{k+1} \approx$$

$$\approx \int_a^A\prod_{i=1}^{k}\frac{A_i-a_i}{n_i}\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_k=1}^{n_k} f\left(x_{i_1}-\frac{h_1}{2}, x_{i_2}-\frac{h_2}{2}, ..., x_{i_k}-\frac{h_k}{2}, x_{k+1}\right)dx_{k+1} \approx$$

$$\approx \frac{A_{k+1}-a_{k+1}}{n_{k+1}}\prod_{i=1}^{k}\frac{A_i-a_i}{n_i}\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_k=1}^{n_k}\sum_{i_{k+1}=1}^{n_{k+1}} f\left(x_{i_1}-\frac{h_1}{2}, x_{i_2}-\frac{h_2}{2}, ..., x_{i_k}-\frac{h_k}{2}, x_{k+1}-\frac{h_{k+1}}{2}\right) =$$

$$= \prod_{i=1}^{k+1}\frac{A_i-a_i}{n_i}\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_k=1}^{n_k}\sum_{i_{k+1}=1}^{n_{k+1}} f_{i_1-1/2,i_2-1/2,...,i_k-1/2,i_{k+1}-1/2}.$$

The obtained result proofs the theory completely.

III. CONCLUSIONS

Thus we have constructed the multi-dimensional cubature formulas to approximate the value of multiple integrals by repeated application of the quadrature formula of trapezoids and rectangles to determine the values of multiple integrals, which are easy to implement on a computer.

REFERENCES

[1]. Demidovich B.L, Maron I.A. "Foundations of Computational Mathematics". - Moscow: Nauka, 1960. - 664 p.
[2]. Rvachev V.L. "Theory of R-functions and some of its applications". - Naukova. Dumka, 1982. - 552 p.
[3]. Rvachev V.L. "An analytical description of some geometric objects". Dokl. Kiev, 1963. - 153, № 4. 765-767pp.
[4]. Nazirov S.A. "Constructive definition of algorithmic tools for solving three-dimensional problems of mathematical physics in the areas of complex configuration" / Iss. Comput. and glue. Mathematics. Tashkent .Scientific works set. IMIT Uzbek Academy of Sciences, 2009. - No. 121. - 15-44pp.
[5]. Kabulov V.K., Faizullaev A., Nazirov S.A. "Al-Khorazmiy algorithm, algorithmization". - Tashkent: FAN, 2006. -664 p.
[6]. Gmurman V.E. "Guide to solving problems in the theory of probability and mathematical statistics". Manual. book for students of technical colleges. - 3rd ed., Rev. and add. - M.: Higher. School, 1979.
[7]. Ermakov S.M. "Monte Carlo methods and related issues". - Moscow: Nauka, 1971.
[8]. Sevastyanov B.A. "The course in the theory of probability and mathematical statistics". - Moscow: Nauka, 1982.
[9]. Mathematics. Great Encyclopedic Dictionary/ Ch. Ed. V. Prokhorov. - Moscow: Great Russian Encyclopedia, 1999.
[10]. Gmurman E. "Probability theory and mathematical statistics." Textbook for technical colleges. 5th ed., revised. and add. - M., Higher. School, 1977.