On rg-Separation Axioms

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Abstract: *In this paper we define almost* rg-*normality and mild* rg-*normality, continue the study of further properties of* rg-*normality. We show that these three axioms are regular open hereditary. Also define the class of almost* rg-*irresolute mappings and show that* rg-*normality is invariant under almost* rg-*irresolute M*-rg-*open continuous surjection.*

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I. Introduction:

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C.E. Aull studied some separation axioms between the T_1 and T_2 spaces, namely, S_1 and S_2. Next, in 1982, S.P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of s-convergence, sequentially semi-closed sets, sequentially s-compact notions. G.B. Navlagi studied P-Normal Almost-P-Normal, Mildly-P-Normal and Pre-US spaces. Recently S. Balasubramanian and P.Aruna Swathi Vyjayanthi studied *v*-Normal Almost- *v*-Normal, Mildly-*v*-Normal and *v*-US spaces. Inspired with these we introduce rg-Normal Almost- rg-Normal, Mildly- rg-Normal, rg-US, rg-S_1 and rg-S_2. Also we examine rg-convergence, sequentially rg-compact, sequentially rg-continuous maps, and sequentially sub rg-continuous maps in the context of these new concepts. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper X and Y denote Topological spaces on which no separation axioms are assumed explicitly stated.

II. Preliminaries:

Definition 2.1: $A\subset X$ is called g-closed[resp: rg-closed] if $clA\subseteq U$[resp: $s$cl(A) $\subseteq$ U] whenever $A\subseteq U$ and U is open[resp: semi-open] in X.

Definition 2.2: A space X is said to be

(i) $T_1(T_2)$ if for $x \neq y$ in X, there exist (disjoint) open sets U; V in X such that $x\in U$ and $y\in V$.

(ii) weakly Hausdorff if each point of X is the intersection of regular closed sets of X.

(iii) Normal [resp: mildly normal] if for any pair of disjoint [resp: regular-closed] closed sets F_1 and F_2 , there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

(iv) almost normal if for each closed set A and each regular closed set B such that $A\cap B = \phi$, there exist disjoint open sets U and V such that $A\subset U$ and $B\subset V$.

(v) weakly regular if for each pair consisting of a regular closed set A and a point x such that $A \cap \{x\} = \phi$, there exist disjoint open sets U and V such that $x \in U$ and $A\subset V$.

(vi) A subset A of a space X is S-closed relative to X if every cover of A by semi-open sets of X has a finite subfamily whose closures cover A.

(vii) R_0 if for any point x and a closed set F with $x\notin F$ in X, there exists a open set G containing F but not x.

(viii) R_1 iff for x, y $\in$ X with $cl\{x\} \neq cl\{y\}$, there exist disjoint open sets U and V such that $cl\{x\}\subset U$, $cl\{y\}\subset V$.

(ix) US-space if every convergent sequence has exactly one limit point to which it converges. (x) pre-US space if every pre-convergent sequence has exactly one limit point to which it converges.

(xi) pre-S_1 if it is pre-US and every sequence pre-converges with subsequence of pre-side points.

(xii) pre-S_2 if it is pre-US and every sequence in X pre-converges which has no pre-side point.

(xiii) is weakly countable compact if every infinite subset of X has a limit point in X.

(xiv) Baire space if for any countable collection of closed sets with empty interior in X, their union also has empty interior in

Definition 2.3: Let $A\subset X$. Then a point x is said to be a

(i) limit point of *A* if each open set containing x contains some point y of *A* such that $x \neq y$.

(ii) T_0–limit point of *A* if each open set containing x contains some point y of *A* such that $cl\{x\} \neq cl\{y\}$, or equivalently, such that they are topologically distinct.

(iii) *pre*-T_0–limit point of *A* if each open set containing x contains some point y of *A* such that $p$cl{x} $\neq$ $p$cl{y}, or equivalently, such that they are topologically distinct.

Note 1: Recall that two points are topologically distinguishable or distinct if there exists an open set containing one of the points but not the other; equivalently if they have disjoint closures. In fact, the T_0–axiom is precisely to ensure that any two distinct points are topologically distinct.

Example 1: Let X = {a, b, c, d} and τ = {{a}, {b, c}, {a, b, c}, X, ϕ}. Then b and c are the limit points but not the T_0–limit points of the set {b, c}. Further d is a T_0–limit point of {b, c}.

Example 2: Let X = (0, 1) and τ = {ϕ, X, and U_n = (0, 1–1⁄n), n = 2, 3, 4,. . . }. Then every point of X is a limit point of X. Every point of X~U_2 is a T_0–limit point of X, but no point of U_2 is a T_0–limit point of X.

Definition 2.4: A set *A* together with all its T_0–limit points will be denoted by T_0–cl*A*.

Note 2: i. Every T_0–limit point of a set *A* is a limit point of the set but converse is not true.
ii. In T_0–space both are same.

Note 3: R_0–axiom is weaker than T_1–axiom. It is independent of the T_0–axiom. However $T_1 = R_0+T_0$

Note 4: Every countable compact space is weakly countable compact but converse is not true in general. However, a T_1–space is weakly countable compact iff it is countable compact.

Definition 3.01: In X, a point x is said to be a rg-T_0–limit point of *A* if each rg-open set containing x contains some point y of *A* such that *rgcl*{x} ≠ *rgcl*{y}, or equivalently; such that they are topologically distinct with respect to rg-open sets.

III. Example

Let X = {a, b, c} and τ = {ϕ, {b}, {a, b}, {b, c}, X}. For A = {a, b}, a is *rg-T_0*–limit point.

Definition 3.02: A set A together with all its rg-T_0–limit points is denoted by T_0-*rgcl* (A)

Lemma 3.01: *If x is a* rg-*T_0–limit point of a set A then x is* rg-*limit point of A.*

Lemma 3.02: *If X is* rg*T_0* [resp: *rT_0*–]–*space then every* rg-*T_0–limit point and every* rg-*limit point are equivalent.*

Theorem 3.03: *For x ≠ y ∈X,*
(i) *X is a* rg-*T_0–limit point of {y} iff x ∉ rgcl{y} and y ∈ rgcl{x}.*
(ii) *X is not a* rg-*T_0–limit point of {y} iff either x ∈ rgcl {y} or rgcl{x} = rgcl{y}.*
(iii) *X is not a* rg-*T_0–limit point of {y} iff either x ∈ rgcl{y} or y ∈ rgcl{x}.*

Corollary 3.04:
(i) *If x is a* rg-*T_0–limit point of {y}, then y cannot be a* rg-*limit point of {x}.*
(ii) *If rgcl{x} = rgcl{y}, then neither x is a* rg-*T_0–limit point of {y} nor y is a* rg-*T_0–limit point of {x}.*
(iii) *If a singleton set A has no* rg-*T_0–limit point in X, then rgclA = rgcl{x} for all x ∈ rgcl{A}.*

Lemma 3.05: *In X, if x is a* rg-*limit point of a set A, then in each of the following cases x becomes* rg-*T_0–limit point of A ({x} ≠ A).*
(i) *rgcl{x} ≠ rgcl{y} for y ∈A, x ≠ y.*
(ii) *rgcl{x} = {x}*
(iii) *X is a* rg-*T_0–space.*
(iv) *A~{x} is* rg-*open*

IV. rg-T_0 AND rg-R_i AXIOMS, i = 0,1:

In view of Lemma 3.5(iii), *rg*-T_0–axiom implies the equivalence of the concept of limit point with that of *rg*-T_0–limit point of the set. But for the converse, if x∈ *rg*cl{y} then *rg*cl{x} ≠ *rg*cl{y} in general, but if x is a *rg*-T_0–limit point of {y}, then *rg*cl{x} = *rg*cl{y}

Lemma 4.01: *In X, a limit point x of {y} is a rg-T_0–limit point of {y} iff rgcl{x} ≠ rgcl{y}.*
This lemma leads to characterize the equivalence of *rg*-T_0–limit point and *rg*-limit point of a set as *rg*-T_0–axiom.

Theorem 4.02: *The following conditions are equivalent:*
(i) *X is a rg-T_0 space*
(ii) *Every rg-limit point of a set A is a rg-T_0–limit point of A*
(iii) *Every r-limit point of a singleton set {x} is a rg-T_0–limit point of {x}*

(iv) For any x, y in X, $x \neq y$ if $x \in rgcl\{y\}$, then x is a rg-T_0–limit point of $\{y\}$

Note 5: In a *rg*-T_0–space X, if every point of X is a r-limit point, then every point is *rg*-T_0–limit point. But if each point is a *rg*-T_0–limit point of X it is not necessarily a *rg*-T_0–space

Theorem 4.03: *The following conditions are equivalent:*
(i) X is a rg-R_0 space
(ii) For any x, y in X, if $x \in rgcl\{y\}$, then x is not a rg-T_0–limit point of $\{y\}$
(iii) A point rg-closure set has no rg-T_0–limit point in X
(iv) A singleton set has no rg-T_0–limit point in X.

Theorem 4.04: *In a rg-R_0 space X, a point x is rg-T_0–limit point of A iff every rg-open set containing x contains infinitely many points of A with each of which x is topologically distinct*

Theorem 4.05: *X is rg-R_0 space iff a set A of the form $A = \cup rgcl\{x_{i\ i=1\ to\ n}\}$ a finite union of point closure sets has no rg-T_0–limit point.*

Corollary 4.06: *The following conditions are equivalent:*
(i) X is a rR_0 space
(ii) For any x, y in X, if $x \in rgcl\{y\}$, then x is not a rg-T_0–limit point of $\{y\}$
(iii) A point rg-closure set has no rg-T_0–limit point in X
(iv) A singleton set has no rg-T_0–limit point in X.

Corollary 4.07: *In an rR_0–space X,*
(i) If a point x is rg-T_0–[resp:*rT_0–*] *limit point of a set then every rg-open set containing x contains infinitely many points of A with each of which x is topologically distinct.*
(ii) If $A = \cup rgcl\{x_{i,\ i=1\ to\ n}\}$ a finite union of point closure sets has no rg-T_0–limit point.
(iii) If $X = \cup rgcl\{x_{i,\ i=1\ to\ n}\}$ then X has no rg-T_0–limit point.

Various characteristic properties of *rg*-T_0–limit points studied so far is enlisted in the following theorem.

Theorem 4.08: *In a rg-R_0–space, we have the following:*
(i) A singleton set has no rg-T_0–limit point in X.
(ii) A finite set has no rg-T_0–limit point in X.
(iii) A point rg-closure has no set rg-T_0–limit point in X
(iv) A finite union point rg-closure sets have no set rg-T_0–limit point in X.
(v) For $x, y \in X$, $x \in T_0$– rgcl$\{y\}$ iff $x = y$.
(vi) $x \neq y \in X$, iff neither x is rg-T_0–limit point of $\{y\}$nor y is rg-T_0–limit point of $\{x\}$
(vii) For any $x, y \in X$, $x \neq y$ iff T_0– rgcl$\{x\} \cap T_0$– rgcl$\{y\} = \phi$.
(viii) Any point $x \in X$ is a rg-T_0–limit point of a set A in X iff every rg-open set containing x contains infinitely many points of A with each which x is topologically distinct.

Theorem 4.09: *X is rg-R_1 iff for any rg-open set U in X and points x, y such that $x \in X\sim U$, $y \in U$, there exists a rg-open set V in X such that $y \in V \subset U$, $x \notin V$.*

Lemma 4.10: *In rg-R_1 space X, if x is a rg-T_0–limit point of X, then for any non empty rg-open set U, there exists a non empty rg-open set V such that $V \subset U$, $x \notin rgcl(V)$.*

Lemma 4.11: *In a rg- regular space X, if x is a rg-T_0–limit point of X, then for any non empty rg-open set U, there exists a non empty rg-open set V such that $rgcl(V) \subset U$, $x \notin rgcl(V)$.*

Corollary 4.12: *In a regular space X, If x is a rg-T_0–[resp: T_0–]limit point of X, then for any $U \neq \phi \in RGO(X)$, there exists a non empty rg-open set V such that $rgcl(V) \subset U$, $x \notin rgcl(V)$.*

Theorem 4.13: *If X is a rg-compact rg-R_1-space, then X is a Baire Space.*
Proof: Routine

Corollary 4.14: *If X is a compact rg-R_1-space, then X is a Baire Space.*

Corollary 4.15: *Let X be a rg-compact rg-R_1-space. If $\{A_n\}$ is a countable collection of rg-closed sets in X, each A_n having non-empty rg-interior in X, then there is a point of X which is not in any of the A_n.*

Corollary 4.16: *Let X be a rg-compact R_1-space. If $\{A_n\}$ is a countable collection of rg-closed sets in X, each A_n having non-empty rg- interior in X, then there is a point of X which is not in any of the A_n.*

Theorem 4.17: *Let X be a non empty compact rg-R_1-space. If every point of X is a rg-T_0–limit point of X then X is uncountable.*

Proof: Since X is non empty and every point is a *rg*-T_0-limit point of X, X must be infinite. If X is countable, we construct a sequence of *rg*-open sets $\{V_n\}$ in X as follows:

Let X = V_1, then for x_1 is a *rg*-T_0-limit point of X, we can choose a non empty *rg*-open set V_2 in X such that $V_2 \subset V_1$ and $x_1 \notin$ *rg*clV_2. Next for x_2 and non empty *rg*-open set V_2, we can choose a non empty *rg*-open set V_3 in X such that $V_3 \subset V_2$ and $x_2 \notin$ *rg*clV_3. Continuing this process for each x_n and a non empty *rg*-open set V_n, we can choose a non empty *rg*-open set V_{n+1} in X such that $V_{n+1} \subset V_n$ and $x_n \notin$ *rg*clV_{n+1}.

Now consider the nested sequence of *rg*-closed sets *rg*cl$V_1 \supset$ *rg*cl$V_2 \supset$ *rg*cl$V_3 \supset \ldots\ldots\ldots \supset$ *rg*cl$V_n \supset \ldots$ Since X is *rg*-compact and {*rg*clV_n} the sequence of *rg*-closed sets satisfies finite intersection property. By Cantors intersection theorem, there exists an x in X such that $x \in$ *rg*clV_n. Further $x \in X$ and $x \in V_1$, which is not equal to any of the points of X. Hence X is uncountable.

Corollary 4.18: *Let X be a non empty rg-compact rg-R_1-space. If every point of X is a rg-T_0–limit point of X then X is uncountable*

V. *rg*–T_0-IDENTIFICATION SPACES AND *rg*–SEPARATION AXIOMS

Definition 5.01: Let $\Re$ be the equivalence relation on X defined by $x\Re y$ iff *rgcl*{x} = *rgcl*{y}

Problem 5.02: show that $x\Re y$ iff *rgcl*{x} = *rgcl*{y} is an equivalence relation

Definition 5.03: $(X_0, Q(X_0))$ is called the rg-T_0–identification space of (X, τ), where X_0 is the set of equivalence classes of $\Re$ and $Q(X_0)$ is the decomposition topology on X_0.

Let P_X: $(X,\tau) \rightarrow (X_0, Q(X_0))$ denote the natural map

Lemma 5.04: *If $x \in X$ and $A \subset X$, then $x \in rgclA$ iff every* rg-*open set containing x intersects A.*

Theorem 5.05: *The natural map $P_X:(X,\tau) \rightarrow (X_0, Q(X_0))$ is closed, open and $P_X^{-1}(P_X(O)) = O$ for all $O \in PO(X,\tau)$ and $(X_0, Q(X_0))$ is* rg-T_0

Proof: Let $O \in PO(X,\tau)$ and $C \in P_X(O)$. Then there exists $x \in O$ such that $P_X(x) = C$. If $y \in C$, then *rgcl*{y} = *rgcl*{x}, which implies $y \in O$. Since $\tau \subset PO(X,\tau)$, then $P_X^{-1}(P_X(U)) = U$ for all $U \in \tau$, which implies P_X is closed and open.

Let G, $H \in X_0$ such that $G \neq H$; let $x \in G$ and $y \in H$. Then *rgcl*{x} ≠ *rgcl*{y}, which implies $x \notin$ *rgcl*{y} or $y \notin$ *rgcl*{x}, say $x \notin$ *rgcl*{y}. Since P_X is continuous and open, then $G \in A = P_X\{X\sim rgcl\{y\}\} \notin PO(X_0, Q(X_0))$ and $H \notin A$

Theorem 5.06: *The following are equivalent:*

(i) X is rgR_0 (ii) $X_0 = \{rgcl\{x\}: x \in X\}$ and (iii) $(X_0, Q(X_0))$ is rgT_1

Proof: (i) ⇒ (ii) Let $x \in C \in X_0$. If $y \in C$, then $y \in$ *rgcl*{y} = *rgcl*{x}, which implies $C \in$ *rgcl*{x}. If $y \in$ *rgcl*{x}, then $x \in$ *rgcl*{y}, since, otherwise, $x \in X\sim rgcl\{y\} \in PO(X,\tau)$ which implies $rgcl\{x\} \subset X\sim rgcl\{y\}$, which is a contradiction. Thus, if $y \in$ *rgcl*{x}, then $x \in$ *rgcl*{y}, which implies *rgcl*{y} = *rgcl*{x} and $y \in C$. Hence X_0 = {*rgcl*{x}: $x \in X$}

(ii)⇒(iii) Let $A \neq B \in X_0$. Then there exists x, $y \in X$ such that A = *rgcl*{x}; B = *rgcl*{y}, and $rgcl\{x\} \cap rgcl\{y\} = \phi$. Then $A \in C = P_X(X\sim rgcl\{y\}) \in PO(X_0, Q(X_0))$ and $B \notin C$. Thus $(X_0, Q(X_0))$ is rg-T_1

(iii) ⇒ (i) Let $x \in U \in RGO(X)$. Let $y \notin U$ and C_x, $C_y \in X_0$ containing x and y respectively. Then $x \notin$ *rgcl*{y}, implies $C_x \neq C_y$ and there exists rg-open set A such that $C_x \in A$ and $C_y \notin A$. Since P_X is continuous and open, then $y \in B = P_X^{-1}(A) \in x \in RGO(X)$ and $x \notin B$, which implies $y \notin$ *rgcl*{x}. Thus $rgcl\{x\} \subset U$. This is true for all *rgcl*{x} implies $\cap rgcl\{x\} \subset U$. Hence *X* is rg-R_0

Theorem 5.07: *(X, τ) is* rg-*R_1 iff $(X_0, Q(X_0))$ is* rg-*T_2*

The proof is straight forward using theorems 5.05 and 5.06 and is omitted

Theorem 5.08: *X is* rg-*T_i; i = 0,1,2. iff there exists a* rg-*continuous, almost–open, 1–1 function from X into a* rg-*T_i space ; i = 0,1,2. respectively.*

Theorem 5.09: *If f is* rg-*continuous,* rg-*open, and x, $y \in X$ such that rgcl{x} = rgcl{y}, then rgcl{f(x)} = rgcl{f(y)}.*

Theorem 5.10: *The following are equivalent*

(i) X is rg-*T_0*

(ii) Elements of X_0 are singleton sets and

(iii)There exists a rg-*continuous,* rg-*open, 1–1 function $f:X \rightarrow Y$, where Y is* rg-*T_0*

Proof: (i) is equivalent to (ii) and (i) ⇒ (iii) are straight forward and is omitted.

(iii) ⇒ (i) Let x, y∈X such that f(x) ≠ f(y), which implies *rgcl*{f(x)} ≠ *rgcl*{f(y)}. Then by theorem 5.09, *rgcl*{x} ≠ *rgcl*{y}. Hence *(X, τ)* is rg-T_0

Corollary 5.11: *X is* rg-T_i *;* $i = 1,2$ *iff X is* rg-T_{i-1} *;* $i = 1,2$*, respectively, and there exists a* rg-*continuous* , rg-*open, 1–1 function* f*:X into a* rg-T_0 *space.*

Definition 5.04: f is point–rg-closure 1–1 iff for x, y∈X such that *rgcl*{x} ≠ *rgcl*{y}, *rgcl*{f(x)} ≠ *rgcl*{f(y)}.

Theorem 5.12: *(i)If* $f:X\rightarrow Y$ *is point–* rg-*closure 1–1 and (X, τ) is* rg-T_0*, then* f *is 1–1*
(ii)If $f:X\rightarrow Y$*, where X and Y are* rg-T_0 *then* f *is point–* rg-*closure 1–1 iff* f *is 1–1*

The following result can be obtained by combining results for rg-T_0– identification spaces, rg-induced functions and rg-T_i spaces; i = 1,2.

Theorem 5.13: *X is* rg-R_i *;* $i = 0,1$ *iff there exists a* rg-*continuous , almost–open point–* rg-*closure 1–1 function* f*: (X, τ) into a* rg-R_i *space;* $i = 0,1$ *respectively.*

VI. rg-Normal; Almost rg-normal and Mildly rg-normal spaces

Definition 6.1: A space X is said to be rg-normal if for any pair of disjoint closed sets F_1 and F_2 , there exist disjoint rg-open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Example 4: Let X = {a, b, c} and τ = {φ, {a}, {b, c}, X }. Then X is rg-normal.

Example 5: Let X = {a, b, c, d} and τ = {φ, {b, d}, {a, b, d}, {b, c, d}, X}. Then X is rg-normal and is not normal.

Example 6: Let X = {a, b, c, d} with τ = {ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, X} is *rg*-normal, normal and almost normal.
We have the following characterization of rg-normality.

Theorem 6.1: For a space X the following are equivalent:
(i) *X* is rg-normal.
(ii) For every pair of open sets *U* and *V* whose union is *X*, there exist rg-closed sets *A* and *B* such that $A\subset U$, $B \subset V$ and $A\cup B = X$.
(iii) For every closed set *F* and every open set *G* containing *F*, there exists a rg-open set *U* such that $F\subset U\subset rgcl(U)\subset G$.
Proof: (i)⇒(ii): Let *U* and *V* be a pair of open sets in a rg-normal space *X* such that $X = U\cup V$. Then $X–U$, $X–V$ are disjoint closed sets. Since X is rg-normal there exist disjoint rg-open sets U_1 and V_1 such that $X–U\subset U_1$ and $X-V\subset V_1$. Let $A = X–U_1$, $B = X–V_1$. Then *A* and *B* are rg-closed sets such that $A\subset U$, $B\subset V$ and $A\cup B = X$.
(ii) ⇒(iii): Let *F* be a closed set and *G* be an open set containing *F*. Then $X–F$ and *G* are open sets whose union is *X*. Then by (b), there exist rg-closed sets W_1 and W_2 such that $W_1 \subset X–F$ and $W_2 \subset G$ and $W_1\cup W_2 = X$. Then $F\subset X–W_1$, $X–G \subset X–W_2$ and $(X–W_1)\cap(X–W_2) = \phi$. Let $U = X–W_1$ and $V= X–W_2$. Then *U* and *V* are disjoint rg-open sets such that $F\subset U\subset X–V\subset G$. As $X–V$ is rg-closed set, we have $rgcl(U) \subset X–V$ and $F\subset U\subset rgcl(U)\subset G$.
(iii) ⇒ (i): Let F_1 and F_2 be any two disjoint closed sets of *X*. Put $G = X–F_2$, then $F_1\cap G = \phi$. $F_1\subset G$ where *G* is an open set. Then by (c), there exists a rg-open set *U* of *X* such that $F_1 \subset U \subset rgcl(U) \subset G$. It follows that $F_2 \subset X–rgcl(U) = V$, say, then *V* is rg-open and $U\cap V = \phi$. Hence F_1 and F_2 are separated by rg-open sets *U* and *V*. Therefore *X* is rg-normal.

Theorem 6.2: A regular open subspace of a rg-normal space is rg-normal.

Definition 6.2: A function f:X→Y is said to be almost–rg-irresolute if for each x in X and each rg-neighborhood V of f(x), *rgcl*(f^{-1}(V)) is a rg-neighborhood of x.
Clearly every rg-irresolute map is almost rg-irresolute.
The Proof of the following lemma is straightforward and hence omitted.

Lemma 6.1: f is almost rg-irresolute iff $f^{-1}(V) \subset$ rg-int(*rgcl*(f^{-1}(V)))) for every V∈*RGO*(Y).

Lemma 6.2: f is almost rg-irresolute iff f(*rgcl*(U)) ⊂ *rgcl*(f(U)) for every U∈*RGO*(X).
Proof: Let U∈*RGO*(X). If y∉*rgcl*(f(U)). Then there exists V∈ *RGO*(y) such that V∩f(U) = ϕ. Hence f^{-1}(V)∩U= ϕ. Since U∈*RGO*(X), we have rg-int(*rgcl*(f^{-1}(V)))∩*rgcl*(U) = ϕ. By lemma 6.1, f^{-1}(V)∩ *rgcl*(U) = ϕ and hence V∩f(*rgcl*(U)) = ϕ. This implies that y∉f(*rgcl*(U)).

Conversely, if V∈*RGO*(Y), then W = X- *rgcl*(f^{-1}(V)))∈ *RGO*(X). By hypothesis, *f*(*rgcl*(W))⊂ *rgcl* (*f*(W))) and hence X- rg-int(*rgcl*(f^{-1}(V))) = *rgcl*(W)⊂f^{-1}(*rgcl*(*f*(W)))⊂*f*(*rgcl*[*f*(X-f^{-1}(V))])⊂ f^{-1}[*rgcl*(Y-V)] = f^{-1}(Y-V) = X-f^{-1}(V). Therefore f^{-1}(V)⊂ rg-int(*rgcl*(f^{-1}(V))). By lemma 6.1, *f* is almost rg-irresolute.

Theorem 6.3: If *f* is M-rg-open continuous almost rg-irresolute, X is rg-normal, then Y is rg-normal.
Proof: Let A be a closed subset of Y and B be an open set containing A. Then by continuity of *f*, f^{-1}(A) is closed and f^{-1}(B) is an open set of X such that f^{-1} (A) ⊂ f^{-1}(B). As X is rg-normal, there exists a rg-open set U in X such that f^{-1}(A) ⊂ U ⊂ *rgcl*(U)⊂ f^{-1}(B). Then *f*(f^{-1}(A))⊂ *f*(U) ⊂ *f*(*rgcl*(U)) ⊂ *f*(f^{-1}(B)). Since *f* is M-rg-open almost rg-irresolute surjection, we obtain A⊂ *f*(U) ⊂ *rgcl*(*f*(U)) ⊂ B. Then again by Theorem 6.1 the space Y is rg-normal.

Lemma 6.3: A mapping *f* is M-rg-closed iff for each subset B in Y and for each rg-open set U in X containing f^{-1}(B), there exists a rg-open set V containing B such that f^{-1}(V)⊂U.

Theorem 6.4: If *f* is M-rg-closed continuous, X is rg-normal space, then Y is rg-normal.
Proof of the theorem is routine and hence omitted.

Theorem 6.5: If *f* is an M-rg-closed map from a weakly Hausdorff rg-normal space X onto a space Y such that f^{-1}(y) is S-closed relative to X for each y∈Y, then Y is rg-T_2.
Proof: Let $y_1 \neq y_2$∈Y. Since X is weakly Hausdorff, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint closed subsets of X by lemma 2.2 [12.]. As X is rg-normal, there exist disjoint V_i ∈RGO(X, $f^{-1}(y_i)$) for i = 1, 2. Since *f* is M-rg-closed, there exist disjoint U_i∈RGO(Y, y_i) and $f^{-1}(U_i) \subset V_i$ for i = 1, 2. Hence Y is rg-T_2.

Theorem 6.6: For a space *X* we have the following:
(a) If *X* is normal then for any disjoint closed sets A and B, there exist disjoint rg-open sets U, V such that A⊂U and B⊂V;
(b) If *X* is normal then for any closed set A and any open set V containing A, there exists an rg-open set U of X such that A⊂U⊂*rgcl*(U) ⊂V.

Definition 6.2: X is said to be almost rg-normal if for each closed set A and each regular closed set B with A∩B = ϕ, there exist disjoint U; V∈RGO(X) such that A⊂U and B⊂V.
Clearly, every rg-normal space is almost rg-normal, but not conversely in general.

Example 7: Let X = {a, b, c} and τ = {φ, {a}, {a, b}, {a, c}, X}. Then X is almost rg-normal and rg-normal.

Theorem 6.7: For a space X the following statements are equivalent:
(i) X is almost rg-normal
(ii) For every pair of sets U and V, one of which is open and the other is regular open whose union is X, there exist rg-closed sets G and H such that G⊂U, H⊂V and G∪H = X.
(iii) For every closed set A and every regular open set B containing A, there is a rg-open set V such that A⊂V⊂*rgcl*(V)⊂B.
Proof: (i)⇒(ii) Let U∈τ and V∈RO(X) such that U∪V = X. Then (X-U) is closed set and (X-V) is regular closed set with (X-U)∩(X-V) = ϕ. By almost rg-normality of X, there exist disjoint rg-open sets U_1 and V_1 such that X-U ⊂ U_1 and X-V ⊂ V_1. Let G = X- U_1 and H = X-V_1. Then G and H are rg-closed sets such that G⊂U, H⊂V and G∪H = X.
(ii) ⇒ (iii) and (iii) ⇒ (i) are obvious.

One can prove that almost rg-normality is also regular open hereditary.
Almost rg-normality does not imply almost rg-regularity in general. However, we observe that every almost rg-normal rg-R_0 space is almost rg-regular.

Theorem 6.8: Every almost regular, *rg*-compact space X is almost rg-normal.

Recall that a function *f* : X→ Y is called rc-continuous if inverse image of regular closed set is regular closed.

Theorem 6.9: If *f* is continuous M-rg-open rc-continuous and almost rg-irresolute surjection from an almost rg-normal space X onto a space Y, then Y is almost rg-normal.

Definition 6.3: X is said to be mildly rg-normal if for every pair of disjoint regular closed sets F_1 and F_2 of X, there exist disjoint rg-open sets U and V such that F_1 ⊂ U and F_2 ⊂ V.

Example 8: Let X = {a, b, c, d} with τ = {ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, X} is Mildly *rg*-normal.

Theorem 6.10: For a space X the following are equivalent.

(i) X is mildly rg-normal.
(ii) For every pair of regular open sets U and V whose union is X, there exist rg-closed sets G and H such that $G \subset U$, $H \subset V$ and $G \cup H = X$.
(iii) For any regular closed set A and every regular open set B containing A, there exists a rg-open set U such that $A \subset U \subset rgcl(U) \subset B$.
(iv) For every pair of disjoint regular closed sets, there exist rg-open sets U and V such that $A \subset U$, $B \subset V$ and $rgcl(U) \cap rgcl(V) = \phi$.

Proof: This theorem may be proved by using the arguments similar to those of Theorem 6.7.

Also, we observe that mild rg-normality is regular open hereditary.

Definition 6.4: A space X is weakly rg-regular if for each point x and a regular open set U containing {x}, there is a rg-open set V such that $x \in V \subset clV \subset U$.

Example 9: Let X = {a, b, c} and τ = {ϕ, {b},{a, b},{b, c}, X}. Then X is weakly *rg*-regular.

Example 10: Let X = {a, b, c} and τ = {ϕ, {a},{b},{a, b}, X}. Then X is not weakly *rg*-regular.

Theorem 6.11: If $f : X \rightarrow Y$ is an M-rg-open rc-continuous and almost rg-irresolute function from a mildly rg-normal space X onto a space Y, then Y is mildly rg-normal.

Proof: Let A be a regular closed set and B be a regular open set containing A. Then by rc-continuity of *f*, $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$. Since X is mildly rg-normal, there exists a rg-open set V such that $f^{-1}(A) \subset V \subset rgcl(V) \subset f^{-1}(B)$ by Theorem 6.10. As *f* is M-rg-open and almost rg-irresolute surjection, $f(V) \in RGO(Y)$ and $A \subset f(V) \subset rgcl(f(V)) \subset B$. Hence Y is mildly rg-normal.

Theorem 6.12: If $f: X \rightarrow Y$ is rc-continuous, M-rg-closed map and X is mildly rg-normal space, then Y is mildly rg-normal.

VII. rg-US spaces:

Definition 7.1: A point y is said to be a
(i) *rg*-cluster point of sequence $<x_n>$ iff $<x_n>$ is frequently in every *rg*-open set containing x. The set of all *rg*-cluster points of $<x_n>$ will be denoted by $rg\text{-}cl(x_n)$.
(ii) *rg*-side point of a sequence $<x_n>$ if y is a *rg*-cluster point of $<x_n>$ but no subsequence of $<x_n>$ *rg*-converges to y.

Definition 7.2: A sequence $<x_n>$ is said to be *rg*-converges to a point x of X, written as $<x_n> \rightarrow^{rg} x$ if $<x_n>$ is eventually in every *rg*-open set containing x.
Clearly, if a sequence $<x_n>$ *r*-converges to a point x of X, then $<x_n>$ *rg*-converges to x.

Definition 7.3: A subset F is said to be
(i) sequentially *rg*-closed if every sequence in F *rg*-converges to a point in F.
(ii) sequentially *rg*-compact if every sequence in F has a subsequence which *rg*-converges to a point in F.

Definition 7.4: X is said to be
(i) *rg*-US if every sequence $<x_n>$ in X *rg*-converges to a unique point.
(ii) rg-S_1 if it is *rg*-US and every sequence $<x_n>$ *rg*-converges with subsequence of $<x_n>$ *rg*-side points.
(iii) rg-S_2 if it is *rg*-US and every sequence $<x_n>$ in X *rg*-converges which has no *rg*-side point.

Definition 7.5: A function *f* is said to be sequentially *rg*-continuous at $x \in X$ if $f(x_n) \rightarrow^{rg} f(x)$ whenever $<x_n> \rightarrow^{rg} x$. If *f* is sequentially *rg*-continuous at all $x \in X$, then *f* is said to be sequentially *rg*-continuous.

Theorem 7.1: We have the following:
(i) Every rg-T_2 space is *rg*-US.
(ii) Every *rg*-US space is rg-T_1.
(iii) X is *rg*-US iff the diagonal set is a sequentially *rg*-closed subset of X x X.
(iv) X is rg-T_2 iff it is both rg-R_1 and *rg*-US.
(v) Every regular open subset of a *rg*-US space is *rg*-US.
(vi) Product of arbitrary family of *rg*-US spaces is *rg*-US.
(vii) Every rg-S_2 space is rg-S_1 and every rg-S_1 space is *rg*-US.

Theorem 7.2: In a *rg*-US space every sequentially *rg*-compact set is sequentially *rg*-closed.

Proof: Let X be *rg*-US space. Let Y be a sequentially *rg*-compact subset of X. Let $<x_n>$ be a sequence in Y. Suppose that $<x_n>$ *rg*-converges to a point in X-Y. Let $<x_{np}>$ be subsequence of $<x_n>$ that *rg*-converges to a point $y \in Y$ since Y is sequentially *rg*-compact. Also, let a subsequence $<x_{np}>$ of $<x_n>$ *rg*-converge to $x \in X\text{-}Y$. Since $<x_{np}>$ is a sequence in the *rg*-US space X, x = y. Thus, Y is sequentially *rg*-closed set.

Theorem 7.3: If *f* and *g* are sequentially *rg*-continuous and Y is *rg*-US, then the set A = {x | *f*(x) = *g*(x)} is sequentially *rg*-closed.

Proof: Let Y be *rg*-US. If there is a sequence $<x_n>$ in A *rg*-converging to $x \in X$. Since *f* and *g* are sequentially *rg*-continuous, $f(x_n) \to^{rg} f(x)$ and $g(x_n) \to^{rg} g(x)$. Hence *f*(x) = *g*(x) and $x \in A$. Therefore, A is sequentially *rg*-closed.

VIII. Sequentially sub-rg-continuity:

Definition 8.1: A function *f* is said to be

(i) sequentially nearly rg-continuous if for each point $x \in X$ and each sequence $<x_n> \to^{rg} x$ in X, there exists a subsequence $<x_{nk}>$ of $<x_n>$ such that $<f(x_{nk})> \to^{rg} f(x)$.

(ii) sequentially sub-rg-continuous if for each point $x \in X$ and each sequence $<x_n> \to^{rg} x$ in X, there exists a subsequence $<x_{nk}>$ of $<x_n>$ and a point $y \in Y$ such that $<f(x_{nk})> \to^{rg} y$.

(iii) sequentially rg-compact preserving if *f*(K) is sequentially rg-compact in Y for every sequentially rg-compact set K of X.

Lemma 8.1: Every function *f* is sequentially sub-rg-continuous if Y is a sequentially rg-compact.

Proof: Let $<x_n> \to^{rg} x$ in X. Since Y is sequentially rg-compact, there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ rg-converging to a point $y \in Y$. Hence *f* is sequentially sub-rg-continuous.

Theorem 8.1: Every sequentially nearly rg-continuous function is sequentially rg-compact preserving.

Proof: Assume *f* is sequentially nearly rg-continuous and K any sequentially rg-compact subset of X. Let $<y_n>$ be any sequence in *f* (K). Then for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $<x_n>$ is a sequence in the sequentially rg-compact set K, there exists a subsequence $<x_{nk}>$ of $<x_n>$ rg-converging to a point $x \in K$. By hypothesis, *f* is sequentially nearly rg-continuous and hence there exists a subsequence $<x_j>$ of $<x_{nk}>$ such that $f(x_j) \to^{rg} f(x)$. Thus, there exists a subsequence $<y_j>$ of $<y_n>$ rg-converging to $f(x) \in f(K)$. This shows that *f*(K) is sequentially rg-compact set in Y.

Theorem 8.2: Every sequentially *s*-continuous function is sequentially rg-continuous.

Proof: Let *f* be a sequentially *s*-continuous and $<x_n> \to^{s} x \in X$. Then $<x_n> \to^{s} x$. Since *f* is sequentially *s*-continuous, $f(x_n) \to^{s} f(x)$. But we know that $<x_n> \to^{s} x$ implies $<x_n> \to^{rg} x$ and hence $f(x_n) \to^{rg} f(x)$ implies *f* is sequentially rg-continuous.

Theorem 8.3: Every sequentially rg-compact preserving function is sequentially sub-rg-continuous.

Proof: Suppose *f* is a sequentially rg-compact preserving function. Let x be any point of X and $<x_n>$ any sequence in X rg-converging to x. We shall denote the set $\{x_n \mid n = 1,2,3, \ldots\}$ by A and $K = A \cup \{x\}$. Then K is sequentially rg-compact since $(x_n) \to^{rg} x$. By hypothesis, *f* is sequentially rg-compact preserving and hence *f*(K) is a sequentially rg-compact set of Y. Since $\{f(x_n)\}$ is a sequence in *f*(K), there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ rg-converging to a point $y \in f(K)$. This implies that *f* is sequentially sub-rg-continuous.

Theorem 8.4: A function $f: X \to Y$ is sequentially rg-compact preserving iff $f_{/K}: K \to f(K)$ is sequentially sub-rg-continuous for each sequentially rg-compact subset K of X.

Proof: Suppose *f* is a sequentially rg-compact preserving function. Then *f*(K) is sequentially rg-compact set in Y for each sequentially rg-compact set K of X. Therefore, by Lemma 8.1 above, $f_{/K}: K \to f(K)$ is sequentially rg-continuous function.

Conversely, let K be any sequentially rg-compact set of X. Let $<y_n>$ be any sequence in *f*(K). Then for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $<x_n>$ is a sequence in the sequentially rg-compact set K, there exists a subsequence $<x_{nk}>$ of $<x_n>$ rg-converging to a point $x \in K$. By hypothesis, $f_{/K}: K \to f(K)$ is sequentially sub-rg-continuous and hence there exists a subsequence $<y_{nk}>$ of $<y_n>$ rg-converging to a point $y \in f(K)$.This implies that *f*(K) is sequentially rg-compact set in Y. Thus, *f* is sequentially rg-compact preserving function.

The following corollary gives a sufficient condition for a sequentially sub-rg-continuous function to be sequentially rg-compact preserving.

Corollary 8.1: If *f* is sequentially sub-rg-continuous and *f*(K) is sequentially rg-closed set in Y for each sequentially rg-compact set K of X, then *f* is sequentially rg-compact preserving function.

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