Composite entire and meromorphic functions and their growth analysis in the light of order and weak type

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Abstract: *In this paper we study the growthproperties of composite entire and meromorphicfunctions whichimprove some earlier results.*

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I. Introduction, Definitions and Notations

We denote by $\mathbb{C}$the set of all finite complex numbers. Let f bea meromorphic function and $g$be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [4] and [10]. In the sequel we use the following notations:

$\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and

$\log^{[0]} x = x$;

and

$\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and

$$\exp^{[0]} x = x .$$

Definition1Theorder ρ_f and lower order $\lambda_f$of an entire function f aredefined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

If f ismeromorphic then

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

The following definition is also well known :

Definition 2 [3] The weak type$\tau_f$of ameromorphicfunction f of finite positive lower order λ_f is defined by

$$\tau_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}.$$

For entire f ,

$$\tau_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \ 0< \lambda_f < \infty.$$

Similarly one can define the growth indicator$\overline{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f as

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}.$$

When f is entire, it can be easily verified that

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty.$$

Definition 3 [9] A function $\rho_f(r)$ iscalled a proximate orderof f relative to T(r,f) if

(*i*) $\rho_f(r)$ *is non-negative and continuous for* $r \geq r_0$ *, say ,*

(*ii*) $\rho_f(r)$ *is differentiable for* $r \geq r_0$ *except possibly at isolated points at which* $\rho_f'(r-0)$ *and* $\rho_f'(r+0)$ *exist ,*

(*iii*) $\lim_{r \to \infty} \rho_f(r) = \rho_f < \infty$,

(*iv*) $\lim_{r \to \infty} r\rho_f'(r)\log r = 0$ *and*

(*v*) $\limsup_{r \to \infty} \frac{T(r,f)}{r^{\rho_f(r)}} = 1.$

In the line of Definition 3 the following definition may be given :

Definition 4 A function $\lambda_f(r)$ is called a lower proximate order of f relative to T(r, f) if

(*i*) $\lambda_f(r)$ *is non-negative and continuous for* $r \geq r_0$ *, say ,*

(*ii*) $\lambda_f(r)$ *is differentiable for* $r \geq r_0$ *except possibly at isolated points at which* $\lambda_f'(r-0)$ *and* $\lambda_f'(r+0)$ *exist ,*

(*iii*) $\lim_{r \to \infty} \lambda_f(r) = \lambda_f < \infty$,

(*iv*) $\lim_{r \to \infty} r\lambda_f'(r)\log r = 0$ *and*

(*v*) $\liminf_{r \to \infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1.$

In the paper we establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions .

II. Lemmas.

In this section we present some lemmas which will be needed in the sequel .

Lemma 1 *[1] Let f be meromorphic and g be entire . Then for all sufficiently large values of r ,*

$$T(r, fog) \leq \{1 + o(1)\}\frac{T(r,g)}{\log M(r,g)}T(M(r,g), f) .$$

Lemma 2 *[2] Let f be meromorphic and g be entire and suppose that* $0 < \mu < \rho_g \leq \infty$ *.Then for a sequence of values of r tending to infinity ,*

$$T(r, fog) \geq T(\exp(r^\mu), f) .$$

Lemma 3 *[6] Let f be meromorphic and g be entire such that* $0 < \mu < \rho_g \leq \infty$ *and* $\lambda_f > 0$ *.Then for a sequence of values of r tending to infinity ,*

$$T(r, fog) \geq T(\exp(r^\mu), g) .$$

Lemma 4 *[5] If f be an entire function then for* $\delta(> 0)$ *the function* $r^{\rho_f + \delta - \rho_f(r)}$ *is ultimately an increasing function of r .*

Lemma 5 *[7] Let f be an entire function. Then for* $\delta(> 0)$ *the function* $r^{\lambda_f + \delta - \lambda_f(r)}$ *is ultimately an increasing function of r .*

III. Theorems.

In this section we present the main results of the paper .

Theorem 1 *Let f , h be any two meromorphic functions and g ,k be any two entire functions such that* $\rho_h < \infty$, $\rho_k < \rho_g$ *and* $\lambda_f > 0$ *.Then*

$$\liminf_{r \to \infty} \frac{\log \{T(r, hok) \log M(r,k)\}}{\log T(r, fog)} = 0 .$$

Proof. As $\rho_g < \rho_k$, we can choose $\varepsilon(> 0)$ in such a way that

$\rho_k + \varepsilon < \rho_g - \varepsilon < \rho_g$.(1)

Now from (1) and Lemma2 it follows that for a sequence of values of r tending to infinity that

$$\log T(r, fog) \geq \log T(\exp r^{(\rho_g - \varepsilon)}, h)$$

$$i.e., \log T(r, fog) \geq (\lambda_f - \varepsilon) \log \exp r^{(\rho_g - \varepsilon)}$$

$i.e., \log T(r, fog) \geq (\lambda_f - \varepsilon)\, r^{(\rho_g - \varepsilon)}$.(2)

Again we have from Lemma1 for all sufficiently large values of r ,

$$T(r, hok) \log M(r, k) \leq \{1 + o(1)\} T(r, k) T(M(r, k), h)$$

$$i.e., \log \{T(r, hok) \log M(r, k)\}$$

$$\leq (\rho_k + \varepsilon) \log r + (\rho_h + \varepsilon) \log M(r, k) + O(1)$$

$$i.e., \log \{T(r, hok) \log M(r, k)\}$$

$\leq (\rho_k + \varepsilon) \log r + (\rho_h + \varepsilon) r^{(\rho_k + \varepsilon)} + O(1)$.(3)

Therefore from (2) and (3) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log \{T(r, hok) \log M(r, k)\}}{\log T(r, fog)} \leq \frac{(\rho_k + \varepsilon) \log r + (\rho_h + \varepsilon) r^{(\rho_k + \varepsilon)} + O(1)}{(\lambda_f - \varepsilon)\, r^{(\rho_g - \varepsilon)}}. \qquad (4)$$

Now in view of (1) it follows from (4) that

$$\liminf_{r \to \infty} \frac{\log \{T(r, hok) \log M(r, k)\}}{\log T(r, fog)} = 0 .$$

This proves the theorem .

Remark 1 *For the validity of Theorem 1, the conditions* $\rho_h < \infty$, $\rho_k < \rho_g$ *and* $\lambda_f > 0$ *are necessary but for meromorphic h with order zero Theorem 1 also holds for* $\rho_g \leq \rho_k$ *which are evident from the following examples :*

Example 1 Let $f = k = \exp z$, $g = \exp(z^2)$ *and* $h = \exp^{[2]} z$.

Then $\lambda_f = 1 > 0$, $\rho_h = \infty$ *and* $\lambda_k = \rho_k = 1 < 2 = \rho_g$.

Now

$$T(r, fog) \leq \log M(r, fog) = \exp(r^2)$$

and $3T(2r, hok) \geq \log M(r, hok) = \exp^{[2]} r$.

So

$$\frac{\log \{T(r, hok) \log M(r, k)\}}{\log T(r, fog)} = \frac{\log T(r, hok) + \log^{[2]} M(r, k)}{\log T(r, fog)}$$

$$\geq \frac{\exp \frac{r}{2} + \log r + O(1)}{r^2}$$

$i.e., \liminf_{r \to \infty} \frac{\log \{T(r, hok) \log M(r, k)\}}{\log T(r, fog)} = \infty$.

Example 2 Suppose $f = h = g = \exp z$ *and* $k = \exp(z^2)$.

Then $\rho_f = \lambda_h = \rho_h = \lambda_g = \rho_g = 1$ *and* $\rho_k = 2$.

Now
$3T(2r, hok) \geq \log M(r, hok) = \exp(r^2) = r^2$

$i.e.,\log T(r,hok)\geq \frac{r^2}{4}+O(1)$.

Also

$T(r,fog)\ \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$.

Therefore

$$\frac{\log\{T(r,hok)\log M(r,k)\}}{\log T(r,fog)}=\frac{\log T(r,hok)+\log^{[2]}M(r,k)}{\log T(r,\ fog)}$$

$$\geq \frac{\frac{r^2}{4}+O(1)+2\log r}{r-\frac{1}{2}\log r+O(1)}\ .$$

$$i.e.,\liminf_{r\to\infty}\frac{\log\{T(r,hok)\log M(r,k)\}}{\log T(r,fog)}=\infty.$$

Example 3 Suppose $f=z$, $g=\exp(z^2)$ *and* $h=k=\exp z$.

Then $\lambda_f=\rho_f=0<\infty$,$\lambda_h=\rho_h=\lambda_k=\rho_k=1<2=\rho_g$.

Therefore

$$T(r,fog)\leq \log M(r,fog)=r^2$$
$$i.e.,\log T(r,fog)\leq 2\log r\ .$$

Also

$T(r,hok)\sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$.

Thus

$$\frac{\log\{T(r,hok)\log M(r,k)\}}{\log T(r,fog)}=\frac{\log T(r,hok)+\log^{[2]}M(r,k)}{\log T(r,\ fog)}$$

$$\geq \frac{r-\frac{1}{2}\log r+\log r+O(1)}{2\log r}$$

$$i.e.,\liminf_{r\to\infty}\frac{\log\{T(r,hok)\log M(r,k)\}}{\log T(r,fog)}=\infty.$$

Example 4 *Let* $f=g=\exp z$, $h=z$ *and* $k=\exp(z^2)$.

Then $\rho_f=\rho_g=1$, $\lambda_h=\rho_h=0$ *and* $\lambda_k=\rho_k=2$.

Now

$$T(r,hok)\leq \log M(r,hok)=\log\exp(r^2)=r^2$$

and $T(r,fog)\sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$.

So

$$\frac{\log\{T(r,hok)\log M(r,k)\}}{\log T(r,fog)}=\frac{\log T(r,hok)+\log^{[2]}M(r,k)}{\log T(r,\ fog)}$$

$$\leq \frac{4\log r}{r-\frac{1}{2}\log r+O(1)}$$

$i.e.,\liminf_{r\to\infty}\frac{\log\{T(r,hok)\log M(r,k)\}}{\log T(r,fog)}=0$.

Example 5 *Let* $f=g=k=\exp z$ *and* $h=z$.

Then $\rho_f=\rho_g=1$, $\lambda_h=\rho_h=0$ *and* $\lambda_k=\rho_k=2$.

Now

$T(r,hok)\leq\ \log M(r,hok)=r$

and $T(r,fog)\sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$.

So

$$\frac{\log\{T(r,hok)\log M(r,k)\}}{\log T(r,fog)}=\frac{\log T(r,hok)+\log^{[2]}M(r,k)}{\log T(r,\ fog)}$$

$$\leq \frac{2\log r}{r-\frac{1}{2}\log r+O(1)}$$

$i.e., \liminf_{r\to\infty} \frac{\log\{T(r,hok)\log M(r,k)\}}{\log T(r,fog)} = 0.$

In the line of Theorem 1 one can easily prove the following theorem :

Theorem 2 *Let* f, h *be any two meromorphic functions and* g, k *be any two entire functions with* $\rho_h < \infty$, $\rho_k < \rho_g$ *and* $\lambda_f > 0$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[2]}\{T(r,hok)\log M(r,k)\}}{\log^{[2]} T(r,fog)} \le \frac{\rho_k}{\rho_g}.$$

The proof is omitted .

In the line of Theorem 2 the following corollary may be deduced :

Corollary 1 *Let* f, h *be meromorphic and* g, k *be entire such that* $\rho_h < \infty$, $\rho_k < \rho_g$ *and* $\lambda_f > 0$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[3]}\{T(r,hok)\log M(r,k)\}}{\log^{[3]} T(r,fog)} \le 1 .$$

Theorem 3 *Let* f, h *be meromorphic and* g, k *be entire such that* $(i)\rho_f < \infty$, $(ii)\ \lambda_h > 0$ $(iii)\lambda_k > 0$, $(iv)\lambda_g < \rho_k$ *and* (v) $0 < \lambda_g < \infty$, $\overline{\tau_g} < \infty$. *Then*

$$\liminf_{r\to\infty} \frac{\log\{T(r,fog)\log M(r,g)\}}{\log T(r,hok)} \le \rho_f \overline{\tau_g} . \min\{\lambda_h^{-1}, \lambda_k^{-1}\} .$$

Proof. By Lemma 1 we obtain for all sufficiently large values of r ,

$$T(r,fog)\log M(r,g) \le \{1+o(1)\}T(r,g)T(M(r,g),f)$$

$$i.e., \log\{T(r,fog)\log M(r,g)\}$$

$$\le (\rho_g+\varepsilon)\log r + (\rho_f+\varepsilon)\log M(r,g) + O(1)$$

$$i.e., \log\{T(r,fog)\log M(r,g)\}$$

$\le (\rho_g+\varepsilon)\log r + (\rho_f+\varepsilon)(\overline{\tau_g}+\varepsilon)r^{\lambda_g} + O(1).$ (5)

Since $\lambda_g < \rho_k$, in view of Lemma 2 it follows for a sequence of values of r tending to infinity that

$$\log T(r,hok) \ge \log T(\exp(r^{\lambda_g}), h)$$

$$i.e., \log T(r,hok) \ge (\lambda_h-\varepsilon)\log\exp(r^{\lambda_g})$$

$i.e., \log T(r,hok) \ge (\lambda_h-\varepsilon)\, r^{\lambda_g}.$ (6)

Similarly in view of Lemma 3 we have for a sequence of values of r tending to infinity

$$\log T(r,hok) \ge \log T(\exp(r^{\lambda_g}), k)$$

$$i.e., \log T(r,hok) \ge (\lambda_k-\varepsilon)\log\exp(r^{\lambda_g})$$

$i.e., \log T(r,hok) \ge (\lambda_k-\varepsilon)\, r^{\lambda_g},$ (7)

where $0 < \varepsilon < \min\{\lambda_h, \lambda_k\}$.

Now from (5) and (6) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log\{T(r,fog)\log M(r,g)\}}{\log T(r,hok)} \le \frac{(\rho_g+\varepsilon)\log r + (\rho_f+\varepsilon)(\overline{\tau_g}+\varepsilon)r^{\lambda_g} + O(1)}{(\lambda_h-\varepsilon)\, r^{\lambda_g}}$$

$i.e., \liminf_{r\to\infty} \frac{\log\{T(r,fog)\log M(r,g)\}}{\log T(r,hok)} \leq \frac{\rho_f \overline{\tau_g}}{\lambda_h}$. (8)

Analogously from (5) and (7) it follows for a sequence of values of r tending to infinity that

$$\frac{\log\{T(r,fog)\log M(r,\ g)\}}{\log T(r,hok)} \leq \frac{(\rho_g+\varepsilon)\log r + (\rho_f+\varepsilon)(\overline{\tau_g}+\varepsilon)r^{\lambda_g} + O(1)}{(\lambda_k-\varepsilon)r^{\lambda_g}}$$

$i.e., \liminf_{r\to\infty} \frac{\log\{T(r,fog)\log M(r,g)\}}{\log T(r,hok)} \leq \frac{\rho_f \overline{\tau_g}}{\lambda_k}$. (9)

Thus the theorem follows from (8) and (9) .

In the line of Theorem 3 one can easily prove the following theorem :

Theorem 4 *Let* f , h *be meromorphic and* g , k *be entire such that* (*i*)$\rho_h < \infty$, (*ii*) $\lambda_f > 0$,(*iii*)$\lambda_g > 0$, (*iv*)$\lambda_k > \rho_g$ *and* (*v*) $0 < \lambda_k < \infty$, $\overline{\tau_k} < \infty$.*Then*

$$\limsup_{r\to\infty} \frac{\log T(r,fog)}{\log\{T(r,hok)\log M(r,k)\}} \geq (\rho_h \overline{\tau_k})^{-1} . \max\{\lambda_f, \lambda_g\} .$$

The proof is omitted .

Theorem 5 *Let* f *be a meromorphic function and* g , h *be two entire functions such that* $\rho_g < \infty$, $\rho_f < \infty$ *and* $\lambda_h > 0$. *Then for any* $\alpha > 1$

$$\liminf_{r\to\infty} \frac{\log T(r,fog)}{\log T(r,hog)} \leq \left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{\rho_f}{\lambda_h}\cdot(4\alpha)^{\rho_g} .$$

Proof. Since $T(r,g) \leq \log^{+} M(r,g)$,we obtain by Lemma 1 for $\varepsilon(>0)$ and for all sufficiently large values of r ,

$$T(r,fog) \leq \{1+o(1)\}T(M(r,g),f)$$

$i.e., \log T(r,fog) \leq (\rho_f+\varepsilon)\log M(r,g) + O(1)$.(10)

For all sufficiently large values of r we know that

$$T(r,hog) \geq \frac{1}{3}\log M\ \{\tfrac{1}{8}M(\tfrac{r}{4},g) + o(1)\ ,\ h\}\ \{\ cf.\ [8]\ \}$$

For $\varepsilon(0 < \varepsilon < \min\{\lambda_h, \lambda_k\})$ we get for all sufficiently large values of r ,

$$\log T(r,hog) \geq (\lambda_h-\varepsilon)\log\{\tfrac{1}{8}M(\tfrac{r}{4},g)+o(1)\} + O(1)$$

$i.e., \log T(r,hog) \geq (\lambda_h-\varepsilon)\log\{\frac{1}{9}M(\frac{r}{4},g)\} + O(1)$

$i.e., \log T(r,hog) \geq (\lambda_h-\varepsilon)\log M(\frac{r}{4},g) + O(1)$

$i.e., \log T(r,hog) \geq (\lambda_h-\varepsilon)\,T(\frac{r}{4},g) + O(1)$. (11)

Since $\varepsilon(>0)$ is arbitrary , it follows from (10) and (11) for all sufficiently large values of r ,

$\liminf_{r\to\infty} \frac{\log T(r,fog)}{\log T(r,hog)} \leq \frac{\rho_f}{\lambda_h} \liminf_{r\to\infty} \frac{\log M(r,g)}{T(\frac{r}{4},g)}$ (12)

Since $\limsup_{r\to\infty} \frac{T(r,g)}{r^{\rho_g(r)}} = 1$, for given ε $(0 < \varepsilon < 1)$ we get for all sufficiently large values of r ,

$T(r,g) < (1+\varepsilon)r^{\rho_g(r)}$ (13)

and for a sequence of values of r tending to infinity

$T(r,g) > (1-\varepsilon)r^{\rho_g(r)}$. (14)

Since for any $\alpha > 1$, $\log M(r,g) \leq \frac{\alpha+1}{\alpha-1} T(\alpha r, g)$, in view of (13) , (14) and for any $\delta(> 0)$ we get for a sequence of values of r tending to infinity that

$$\frac{\log M(r,g)}{T(\frac{r}{4},g)} \leq \frac{\frac{\alpha+1}{\alpha-1}(1+\varepsilon)}{(1-\varepsilon)} \cdot \frac{(\alpha r)^{\rho_g+\delta}}{(\alpha r)^{\rho_g+\delta-\rho_g(\alpha r)}} \cdot \frac{1}{(\frac{r}{4})^{\rho_g(\frac{r}{4})}}$$

$$\leq \left(\frac{\alpha+1}{\alpha-1}\right)\frac{(1+\varepsilon)}{(1-\varepsilon)} \cdot \frac{(\frac{4\alpha r}{4})^{\rho_g+\delta}}{(\frac{4\alpha r}{4})^{\rho_g+\delta-\rho_g(\frac{4\alpha r}{4})}} \cdot \frac{1}{(\frac{r}{4})^{\rho_g(\frac{r}{4})}}$$

$$\leq \left(\frac{\alpha+1}{\alpha-1}\right)\frac{(1+\varepsilon)}{(1-\varepsilon)} \cdot (4\alpha)^{\rho_g+\delta}$$

because$r^{\rho_g+\delta-\rho_g(r)}$ is ultimately an increasing function of$r$. Since $\varepsilon(> 0)$ and $\delta(> 0)$ are arbitrary , we obtain that

$$\liminf_{r\to\infty} \frac{\log M(r,g)}{T(\frac{r}{4},g)} \leq \left(\frac{\alpha+1}{\alpha-1}\right) \cdot (4\alpha)^{\rho_g} . \qquad (15)$$

Thus from (12) and (15) it follows that

$$\liminf_{r\to\infty} \frac{\log T(r,fog)}{\log T(r,hog)} \leq \left(\frac{\alpha+1}{\alpha-1}\right) \cdot \frac{\rho_f}{\lambda_h} \cdot (4\alpha)^{\rho_g} .$$

In the line of Theorem 5 one can easily prove the following theorem using the definition of lower proximate order :

Theorem 6 *Let f be a meromorphic function and g , h , k be any three entire functions such that $\rho_g < \lambda_k < \infty$ and$\lambda_h < \infty$ .Then for any $\alpha > 1$*

$$\liminf_{r\to\infty} \frac{\log T(r,fog)}{\log T(r,hog)} \leq \left(\frac{\alpha+1}{\alpha-1}\right) \cdot \frac{\rho_f}{\lambda_h} \cdot (4\alpha)^{\lambda_g} .$$

The proof is omitted .

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